A COVARIANCE-INVARIAT COARSE-SAMPLED-DATA FILTER.

by

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— Geeta Wattal

CERTIFICATE

This is to certify that the thesis entitled 'A Covariance-Invariant Coarse-Sampled-Data Filter' by Geeta Wattal is a record of work carried out under my supervision and has not been submitted elsewhere for a degree.

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ABSTRACT

Linear Time Invariant digital filters are characterized by difference equations with constant coefficients. They generally require complex hardware structures when they are realised physically. This thesis deals with the development of a non-linear digital filter that is equivalent to a linear digital filter, in the sense that the expected values of their outputs are equal. It is shown that for a fixed input sampling rate, the power spectrum of the output of the nonlinear filter can be made to asymptotically approach the power spectrum of the output of the linear filter. The hardware requirement for the nonlinear filter is simpler than that required for the linear filter. A possible application of this filter structure is in the realisation of filter banks, and in the realisation of complex filters, such as a fading dispersive channel simulater.

CONTENTS

CHAPTER 1

INTRODUCTION

Digital signal processing techniques find increasing application over analog signal processing techniques owing to their greater accuracy, reproduciblity and programmability. Advances in integrated circuit technology have made, for the same accuracy, dsp techniques more economical than analog signal processing techniques. The theory of dsp emerged from an understanding of sampling and its spectral effects and the use of the z-transform. The integrated circuit implementation of digital circuits caused dsp to further evolve a and it continues to do so today.

The sampling of a signal is quantization of the signal in time. Similarly the concept of the quantization of the signal in amplitude developed. The late 40's saw the introduction of the statistical theory of amplitude quantization. One of the numerous uses of quantizers was in correlation devices. Digital signal processing used finite wordlength machines, which necessarily meant quantization in amplitude. The statistical theory of amplitude quantization was applied to study the finite wordlength effects in digital machines and, in dsp areas, specifically in digital filters.

Linear digital filtering is one area of digital signal processing, where the digital filter is the implementation of a constant coefficient linear difference equation. The hardware realisation of a digital filter therefore requires a number of basic hardware units e.g. multipliers,delay elements, and summers, the number of elements increasing as the order of the filter, hence increasing the cost. Kirlin [7] introduced the statistical coarse-sampled-data filter with a view to reduce the hardware requirements of the linear filter. It was his contention that the performance of the coarse-sampled-data filter matched that of the linear digital filter it meant to replace.

Kirlin's nonlinear filter appears to be very promising, for while it is said to be equal to the linear digital filter in its performance, its hardware requirements are simpler, implying a reduction in hardware complexity as well as cost. This aspect is particularly important in the realisation of higher order systems. For e.g., a digital hardware unit may simulate a fading dispersive channel for eg, a troposcatter channel. The large bank of filters required to adequately simulate multipath delays, and the multiplication by complex coefficients to simulate Rayleigh fading and Doppler Spread, require a large amount

of complex hardware. Implementing a fading dispersive channel simulator using Kirlin's scheme would mean a big saving in hardware.

The object of the present work was to evaluate the performance of Kirlin's coarse-sampled-data filter with respect to that of the linear digital filter. If the performance of the coarse-sampled-data filter was found to be not equal to that of the linear digital filter, could the structure of the coarse-sampled-data filter be changed so that the performance of the modified filter matched closely, that of the linear digital filter?

It was found that the performance of the coarse-sampled data filter when the input was a white noise sequence, was nowhere near that of the linear digital filter. However, altering the structure of the feedback of the coarse-sampled-data filter led to what has been called 'N-bit feedback sampled-data filter'. The performance of the output of the N-bit feedback sampled-data filter, when the input was a white noise sequence, was found to asymptotically approach that of the linear digital filter.

Organization of Chapters:

In Chapter 2, theory and techniques of digital filtering, the statistical theory of quantization and the

coarse-sampled-data filter using an RC averager, are reviewed. In Chapter 3, the performance of the first order and second order linear digital filters, coarse-sampled-data filters and N-bit feedback sampled-data filters are evaluated, for a white noise sequence input. In Chapter 4, the performance of first and second order N-bit feedback sampled-data filters as obtained by simulating the filters on the DEC-10 is discussed. Chapter 5 summarises the results obtained in Chapter 3 and 4.

Chapter 2

REVIEW OF DIGITAL SIGNAL PROCESSING AND THE STATISTICAL THEORY OF QUANTISATION

In this chapter, digital filter design techniques, effects of quantization on analog signals, use of quantizers for autocorrelation determination, and the statistical coarse sampled-data filter are discussed.

2.1 RELATION BETWEEN ANALOG AND DISCRETE SYSTEMS

The Fourier transform of an analog signal $x_A(t)$ is given by

$$X_A(j\omega) = \int_{-\infty}^{\infty} x_A(t)\, e^{-j\omega t}\, dt \tag{2.1.1}$$

Inversely $x_A(t)$ is derived from $X_A(j\omega)$ by the relation

$$x_A(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_A(j\omega)\, e^{j\omega t}\, d\omega \tag{2.1.2}$$

If $x_A(t)$ be sampled, with sampling rate $1/T$, it can be shown [1 that for the resultant signal $x(nT)$, the Fourier transform is given by

$$X(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_A(j\omega - kW_s) \tag{2.1.3}$$

where ω_s is the sampling frequency in radians and $\omega_s = 2\pi/T$. $X(j\omega)$ is obviously periodic with period $2\pi/T$.

If $X_A(j\omega)$ is band limited over the range $(-\pi/T, \pi/T)$, then $X_A(j\omega)$ and $X(j\omega)$ are identical in this frequency range. Otherwise a condition known as 'aliasing' occurs.

When $x_A(t)$ is a stationary stochastic process, it can be shown [2], that the ensemble autocorrelation function $R_x(l)$ of the sequence $x(nT)$ and the ensemble autocorrelation function $R_{x_A}(\tau)$ of the analog signal $x_A(t)$ are related by

$$R_x(l) = R_{x_A}(\tau) \Big|_{\tau=lT} \qquad (2.1$$

It can also be shown [2] that if $x_A(t)$ is ergodic in the Autocorrelation sense then the time averaged autocorrelation function of $x_A(t)$, defined by

$$R_{x_A}^*(\tau) = \lim_{K \to \infty} \frac{1}{2K+1} \int_{-K}^{K} x_A(t)\, x_A(t+\tau)\, dt$$

and the time averaged autocorrelation function $R_x^*(l)$ of the sampled signal $x(nT)$ defined in like manner, are related as follows :

$$R_x^*(l) = R_{x_A}^*(\tau) \Big|_{\tau=lT} \qquad (2.1.5)$$

The power spectrum and the ensemble ACF of $x_A(t)$ form a Fourier transform pair thus :

$$S_{x_A}(j\omega) \leftrightarrow R_{x_A}(\tau)$$

and

$$S_{x_A}(j\omega) = \int R_{x_A}(\tau)\, e^{-j\omega\tau}\, d\tau \qquad (2.1.6)$$

The power spectrum of the sampled signal x(nT) is given by

$$S_x(j\Omega T) = S_x(z)\Big|_{z=e^{j\Omega T}} = \sum_{k=-\infty}^{\infty} R_x(k)\, z^{-k}\Big|_{z=e^{j\Omega T}} \qquad (2.1.7)$$

$$\mathcal{L}\,[R_x(m)\ \delta(\ \ T)]$$

Eqn. (2.1.4) effectively implies that $R_x(1)$ may be obtained by sampling $R_{x_A}(\tau)$ at T second intervals, and because $R_{x_A}(\tau) \leftrightarrow S_{x_A}(j\omega)$

$$\therefore\ S_x(j\omega T) \triangleq \mathcal{L}\,[R_x(k)] = \frac{1}{T} \sum_{m=-\infty}^{\infty} S_{x_A}(j\overline{\omega - W_s}\,T) \qquad (2.1.8)$$

by involving the relation (2.1.3).

2.2 DIGITAL FILTER DESIGN TECHNIQUES

2.2.1 Filter Properties :

Important properties of digital filters are magnitude squared response $|H(e^{j\omega})|^2$, phase delay $T_p(\omega)$, and group or envelope delay $T_g(\omega)$.

If the filter transfer function is given by H(z), then

$$|H(e^{j\omega})|^2 = [H(z)\ H(z^{-1})]\Big|_{z=e^{j\omega}} \qquad (2.2.1)$$

$$T_p(j\omega) = \mathrm{Tan}^{-1}\left[\frac{I_m(H(z))}{R_e(H(z))}\Big|_{z=e^{j\omega}}\right] \qquad (2.2.2)$$

$$= \frac{1}{2j}\ \ln\left[\frac{H(z)}{H(z^{-1})}\Big|_{z=e^{j\omega}}\right] \qquad (2.2.3)$$

$$T_g(j\omega) = -\frac{d}{d} T_p(j\omega) = -jz \frac{dT_p(z)}{dz}\Bigg|_{z=e^{j\omega}} \quad (2.2.4)$$

$$T_g(j\omega) = -\mathrm{Re}[z(\frac{d\ H(z)}{dz}) \frac{1}{H(z)}]\Bigg|_{z=e^{j\omega}} \quad (2.2.5)$$

$$= -\mathrm{Re}[z \frac{d}{dz} (\ln H(z))]\Bigg|_{z=e^{j\omega}} \quad (2.2.6)$$

Those poles of $H(e^{j\omega})^{2}$ that lie inside the unit circle in the z-plane are determined to be the poles of H(z). The zeros of H(z) are not uniquely determined, but choosing them to be those zeros of $H(e^{j\omega})^{2}$ that lie inside or on the unit circle, yields a minimum phase filter. Some signal processing applications require the filter to be linear phase i.e. that the group delay be zero or a constant over the frequency bands of interest.

2.2.2 Finite Duration Impulse Response Filters :

Two widely used filter types are FIR and IIR filters. Some advantages of FIR filters are that they are always stable and using an appropriate delay can always be realized. They can be designed to yield exactly linear phase, a condition required for speech signal processing and data transmission. Disadvantages of FIR filters are that a large value of impulse response duration is required to adequately approximate sharply cut off filters, hence requiring a large amount of computation and time. But with the advent of the FFT, fast convolution has been made possible, so that FIR filters are now competitive

with sharp cut-off IIT filter designs. If an FIR filter with impulse response duration = N, be required to be linear phase with $T_p(j\omega) = -\alpha$ then

$$\alpha = \frac{N-1}{2} \tag{2.2.7}$$

and $$h(n) = h(N-1-n) \tag{2.2.8}$$

where $\{h(n),\ n = 0,\ \ldots,\ N-1\}$ is the impulse response sequence.

FIR Filter design techniques :

(a) Windowing : The Fourier Transform of the impulse response of a digital filter is periodic and it can be expanded in a Fourier series thus :

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)\ e^{-j\omega n} \tag{2.2.9}$$

where $\{h(n)\}$ is the impulse response of the filter and given by

$$h(n) = \frac{1}{2\pi} \int_0^{2\pi} H(e^{j\omega})\ e^{j\omega n}\ d\omega\ ,\ -\infty \leq n \leq \infty \tag{2.2.10}$$

Because $\{h(n)\}$ extends from $-\infty \leq n \leq \infty$, the filter is unrealizable. A finite weighting function w(m) is used to modify the Fourier coefficients $\{h(n)\}$ so that $\hat{h}(n) = w(n).h(n)$. Then if $\hat{H}(e^{j\omega}) \leftrightarrow \hat{h}(n)$, $\hat{H}(e^{j\omega})$ should

approximate $H(e^{j\omega})$ within some permissible error. Clearly $\hat{H}(e^{j\omega})$ is the circular convolution of $W(e^{j\omega})$ and $H(e^{j\omega})$. If $W(e^{j\omega}) = \delta(\omega)$ then $\hat{H}(e^{j\omega}) = H(e^{j\omega})$, but if $w(n)$ is a finite sequence, $W(e^{j\omega})$ is not an impulse. Then $W(e^{j\omega})$ must be chosen such that its main lobe has a small width and hence higher energy concentration; the side lobes should decrease in energy rapidly as $\omega \to \pi$. Some windows used are Rectangular window, Hamming window, Kaiser window.

(b) Frequency Sampling : Given N equally spaced points $\{H(k)\}$ in the DFT of an impulse response sequence, the transfer function $H(z)$ can be found [1, pp 105-107] to be

$$H(z) = \frac{1-z^{-N}}{N} \sum_{k=0}^{N-1} \frac{H(k)}{(1-z^{-1}\exp(-\frac{2\pi k}{N}))} \tag{2.2.10}$$

Here the transfer function $H(z)$ approximates a continuous transfer function. The approximation error is zero at the N specified frequencies and finite between them.

(c) Optimal (Min Max Error) Filters : Considering the linear phase FIR filter design problem as a Chebyshev Approximation problem, a set of condition is derived for which it can be proved that the solution for which the peak approximation error over the interval of approximation is minimized, is optimal. Define $H_o(e^{j\omega T})$ = the transfer fn. being approximated, $W(e^{j\omega T})$ = a weighting function on the approximation error, so that the relative size of the error in different

frequency bands may be chosen, $H(e^{j\omega T}) \triangleq$ the frequency response of the designed filter. Then

$$E(e^{j\omega T}) = W(e^{j\omega T}) \left[H_o(e^{j\omega T}) - H(e^{j\omega T})\right] \qquad (2.2.12)$$

where $E(e^{j\omega T})$ is the approximation error. Then the Chebyshev approximation problem may be stated as

$$E(e^{j\omega T})\Big|_{min} = \min_{coeffs} \left[\max_{\omega \varepsilon A} E(e^{j\omega T})\right] \qquad (2.2.13)$$

where A is the disjoint union of frequency bands of inte

Of these three design technique, the frequency sampling and Minmax error techniques are superior to windowing.

2.2.3 IIR Filter Design : All filter design techniques should lead to stable and realizable filters, therefore IIR filters should be such that

$$\begin{aligned} h(n) &= 0, & n < 0 \\ &= h(n) & n > 0 \end{aligned} \qquad (2.2.14)$$

The general form of the IIR filter transfer function is

$$H(z) = \left(\sum_{i=0}^{M} a_i z^{-i}\right) / \left(1 + \sum_{i=1}^{N} b_i z^{-i}\right) \qquad (2.2.15)$$

where at least one b_i is nonzero and the roots of the denominator are not cancelled by roots of the numerator, which case

leads to an FIR filter. For $H(z)$ derived from an analog filter, $M \leq N$.

Design Techniques :

Impulse Invariant Transformation : Using this transformation, the impulse response of the resulting digital filter is a sampled version of that of an analog filter. Let $H(s)$ have simple poles and $M \leq N$, then

$$H(s) = \sum_{i=1}^{N} \frac{C_i}{s+d_i} \; ; \quad C_i = H(s)\,(s+d_i)\Big|_{s=-d_i} \qquad (2.2.16)$$

The analog filter impulse response is given by

$$h(t) = \sum_{i=1}^{N} C_i \, e^{-d_i t} \cdot u_{-1}(t) \qquad (2.2.17)$$

$$\therefore \; h(nT) = \sum_{i=1}^{N} C_i \, e^{-d_i t} \cdot u_{-1}(nT) \qquad (2.2.18)$$

where T is the sampling period.

$$H(z) = \sum_{i=1}^{N} h(nT)\, z^{-n} = \sum_{i=1}^{N} \frac{c_i}{1-e^{-d_i T} z^{-1}} \qquad (2.2.19)$$

Obviously, this uses the mapping $s+d_i \longrightarrow \left(1-e^{-d_i T} z^{-1}\right)$ for a simple pole at $-d_i$. Since $H(z=e^{j\omega T})$ is the Fourier Transform of the sampled signal $h(t)$.

$$H(e^{j\omega T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H(j\overline{\omega - k\,w_s}) \;, \text{ where } w_s = 2\pi/T.$$

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$$H(e^{j\omega T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H(j\overline{\omega - k\, w_s}) \text{ , where } w_s = 2\pi/T.$$

The equivalent analog filter must be bandlimited to the range $(-\pi/T, \pi/T)$; as T becomes larger the effect of aliasing becomes negligible and the digital and analog frequency responses become comparable.

Bilinear Transformation : Using this technique, analog filters are converted to equivalent digital filters using a transformation [3] that maps the s-plane into the z-plane. The bilinear transformation is given by

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \tag{2.2.20}$$

in the z-plane. The LH-s plane and the RH s-plane are mapped inside and outside the unit circle in the z-plane respectively. On the unit circle $z = e^{j\Omega T}$

$$j\omega = \frac{2}{T}\,\frac{(1-e^{-j\Omega T})}{(1+e^{-j\Omega T})} = \frac{2}{T}\tan\left(\frac{\Omega T}{2}\right) \tag{2.2.21}$$

where ω and Ω are the corresponding analog and digital frequencies, and their relation is highly nonlinear. Therefore this transformation can be used only when the analog filter response is piecewise constant. Though realizable, stable analog filters are mapped to realizable digital filters, the nonlinear relation between ω and Ω leads to a highly warped digital filter response.

Matched z Transformation : The poles and zeros in the s-plane are mapped onto the poles and zeros in the z-plane, where the mapping is defined by $s+a \rightarrow 1-z^{-1}e^{-aT}$ where T is the sampling period. The matched z-transformation is not a suitable mapping when the analog filter has zeros at centre frequencies greater than half the sampling frequency, as the position of the zero will be greatly aliased in the z-plane.

Least Squares Error Method :

Assume the IIR filter's transfer function is of the form

$$H(z) = A \prod_{k=1}^{k} \frac{1+a_k z^{-1} + b_k z^{-2}}{1+c_k z^{-1} + d_k z^{-2}} \qquad (2.2.22)$$

Let the desired frequency response be $[H_o(e^{j\omega_i}),\ i=1,2,3,\ \ldots,M]$ where ω_i are not necessarily equally spaced. The squared error in frequency is $Q(\theta)$

$$Q(\theta) = \sum_{i=1}^{M} \left[H(e^{j\omega_i}) - H_o(e^{j\omega_i})\right]^2 \qquad (2.2.23)$$

where

$$\theta = (a_1, b_1, c_1, d_1, \ldots, a_k, b_k, c_k, d_k, A) \qquad (2.2.24)$$

$Q(\theta)_{min}$ is found using methods of calculus on the computer. [1] provides an exhaustive study of FIR and IIR filter design techniques.

2.3 QUANTIZATION EFFECTS IN DIGITAL SYSTEMS

The analysis of digital systems is based on the theory of difference equations with constant coefficients. The wordlength of any digital system is finite. Therefore quantization of the input to the system and discretization of the coefficients are compulsory. Quantization leads to approximation of the signal value by one of an appropriate set of discrete levels not from a continuous range. This nonlinearization makes an analysis of the system difficult except for simple cases. A statistical analysis may however be carried out to determine the resultant distortion.

Quantization leads to rounding or 'grouping' of input data samples, the range of each sample is subdivided into 'intervals' of equal width q.

$$i_q - \frac{q}{2} < x \leq i_q + \frac{q}{2} \quad i = 0, \pm 1, \pm 2 \ldots \tag{2.3.1}$$

The nonlinear quantization replaces each value of x by the nearest interval centre 'i_q'. The quantiser output x' can be considered the sum of the input x and a round-off error referred to as 'quantisation noise' x_q

$$x' = x + x_q \tag{2.3.2}$$

The noise x_q may increase the variance of x' and bias the mean of the estimate x', but for an important class of random process

it has been shown that [11] the biasing effects of quantization average out approximately or can be predicted and corrected even with coarse quantization. The statistical theory of Amplitude Quantization was put forward by Bennet [12] but is essentially due to Wideow [11]. His principle results are as follows :

Let x,y be two random variables with probability density functions (p.d.f) $p(x)$, $p(y)$ and joint density function $p(x,y)$. Let the characteristic functions be $W_x(\alpha) = E(e^{j\alpha x})$ and $W_{x,y}(\alpha_1,\alpha_2) = E[\exp(j(\alpha_1 x + \alpha_2 y))]$.

The quantization theorems are defined as follows :

1. If $p(x)$ is 'bandlimited' so that $W_x(\alpha)$ vanishes for $|\alpha| \geqslant \frac{2\pi}{q} - \varepsilon$ $(\varepsilon > 0)$, then every existing mean $E[x^m]$ is completely determined by $E[(x')^m]$ and the first order probability density function of x_q is uniform between $(-q/2, q/2)$.
2. If $p(x,y)$ is 'bandlimited' so that $W_{x,y}(\alpha_1,\alpha_2)$ vanishes for $|\alpha_1| \geqslant \frac{2\pi}{q} - \varepsilon$, $|\alpha_2| \geqslant \frac{2\pi}{q} - \varepsilon$, $\varepsilon > 0$, then every existing moment $E\, x^m y^n$ $(m,n = 1,2, \ldots)$ is given by $E((x')^m (y')^n)$ and x_q and y_q are uniformly distributed and statistically independent.

Widrow also gives a rigorous justification of the relations known as 'Sheppard's corrections'.

$$E(x) = E(x'); \quad E(x^2) = E((x')^2) - \frac{1}{12} q^2 \qquad (2.3.3)$$

Watts [14] has extended Widrow's theory to the case of quantization by shifted class intervals, i.e.

$$\left.\begin{aligned} a_1 + i\,q_1 - q_1/2 &< x \leq a_1 + i\,q_1 + q_1/2 \\ a_2 + k\,q_2 - q_2/2 &< y \leq a_2 + k\,q_2 + q_2/2 \end{aligned}\right\} \quad (2.3.4)$$

The input-output characteristic of the quantizer (x' vs x) and its composition into shifters, gains and unit quantiser, are shown in Fig. (2.3.1). The p.d.f. $p(x')$ of the output of the general quantiser is

$$p(x') = \frac{1}{r} \sum_{n=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} p(x) f_1 \left[\frac{x'}{r} - c - \left(\frac{x}{q} - a\right)\right] dx\right] . \delta\left(\frac{x'}{r} - c - n\right) \quad (2.3.5)$$

$f_1(x)$ is a window function s.t. $f(x) = 1$ for $-\frac{1}{2} < x \leq \frac{1}{2}$, $f(x) = 0$ elsewhere. $\delta(x)$ is the Dirac delta function.

The characteristic function (c.f) of x' is

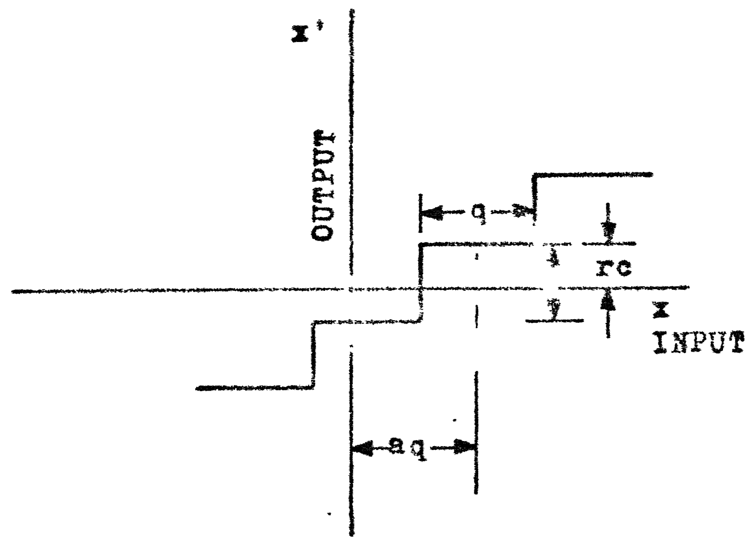
$$W_{x'}(\alpha) = \sum_{k=-\infty}^{\infty} \exp(-j\alpha(c-a)) \exp(-j2\pi ka)\, W_x\left(\frac{\alpha r}{q} - \frac{2\pi k}{q}\right) \frac{\sin \frac{1}{2}(\alpha r - 2\pi k)}{\frac{1}{2}(\alpha r - 2\pi k)} \quad (2.3.6)$$

If x and y, with joint p.d.f $p(x,y)$ are each quantised by general quantisers with parameters (a_i, q_i, c_i, r_i), $i = 1, 2, \ldots,$ the resulting c.f. of x' and y' is $W_{x',y'}(\alpha_1, \alpha_2)$,

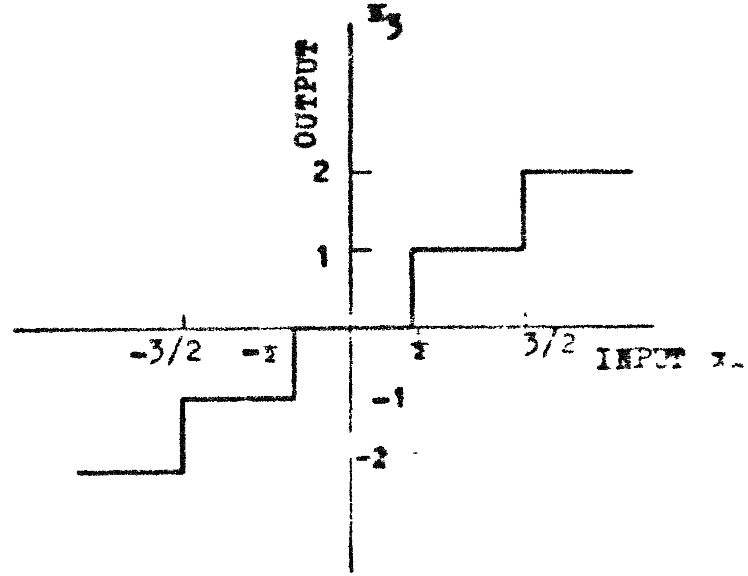
$$W_{x',y'}(\alpha_1,\alpha_2) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \exp(-jr_1\alpha_1(c_1-a_1)).\exp(-j\alpha_2 r_2(c_2-a_2)).$$

$$\exp(-j2\pi k a_1) \exp(-j2\pi l a_2) . W_{x,y}\left(\frac{\alpha_1 r_1}{q_1} - \frac{2\pi k}{q_1}, \frac{\alpha_1 r_2}{q_2} - \frac{2\pi l}{q_2}\right).$$

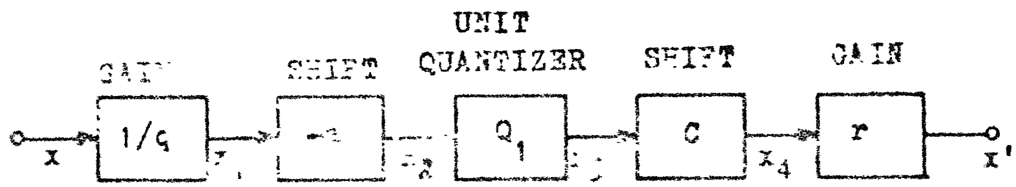
$$\frac{\sin\frac{1}{2}(\alpha_1 r_1 - 2\pi k)}{\frac{1}{2}(\alpha_1 r_1 - 2\pi k)} \cdot \frac{\sin \frac{1}{2}(\alpha_2 r_2 - 2\pi l)}{\frac{1}{2}(\alpha_2 r_2 - 2\pi l)} \quad (2.3.7)$$



(a) General Quantizer



(b) Unit Quantizer Q_1



(c) Composition of the General Quantizer

Fig. 2.3.1

This result may be extended to any number of jointly related variables. Quantization has previously been described as the addition of independent noise to the signal input to the quantiser, i.e.

$$x' = x + x_q$$

then $E(x'y') = E(\overline{x+x_q} \cdot \overline{y+y_q}) = E(xy + x_q y + x y_q + x_q y_q)$ (2.3.8)

In terms of the c.f.

$$E((x')^m (y')^n) = \frac{1}{(-j)^{m+n}} \left[\frac{\partial^{m+n} W_{x',y'}(\alpha_1,\alpha_2)}{\partial \alpha_1^m \, \partial \alpha_2^n} \right]_{\alpha_1=\alpha_2=0} \quad (2.3.9)$$

Let their respective means be subtracted from x and y, and $W_o = W_o\left(\frac{\alpha_1 r_1}{q_1} - \frac{2\pi k}{q_1}, \frac{\alpha_2 r_2}{q_2} - \frac{2\pi \ell}{q_2}\right)$ be the c.f of $p(x-\bar{x}, y-\bar{y})$, it is shown [14, p 211] that if the c.f. $W_o(\alpha_1,\alpha_2)$ be zero for $|\alpha_1| \geq \frac{2\pi}{q_1}$ and $|\alpha_2| \geq \frac{2\pi}{q_2}$ then

$$E(x'y') = E(xy) \quad (2.3.10)$$

i.e., the quantisation noises are uncorrelated with both the input signals and with each other.

Use of Dither :

If a random variable n satisfies the first & second quantizing theorems, and if x and n are statistically independent then the sum x+n also satisfies the quantizing theorems, regardless of the distribution of x. Chang and Moore [16] studied the modified digital correlator shown in Fig. 2.3.2.

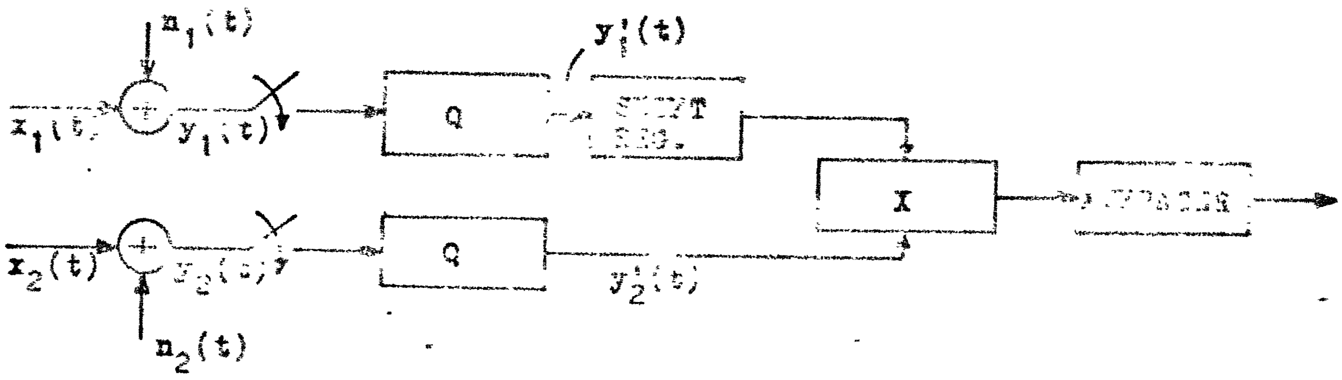


Fig. 2.3.2 A Modified Digital Correlator.

Let the c.f. of $n_1(t)$ and $n_2(t)$ be $W_{n_1}(\alpha_1)$ and $W_{n_2}(\alpha_2)$ respectively. Chang and Moore [16] show that the correlation function of the outputs of the quantizers $y_1'(t)$ and $y_2'(t)$ and that of the arbitrary inputs $x_1(t)$ and $x_2(t)$ are equal iff the auxiliary noise waveforms $n_1(t)$, $n_2(t)$ are such that

$$W_{n_1}\left(-\frac{2\pi k}{q_1}\right) = 0 \qquad k \neq 0$$

and (2.3.11)

$$W_{n_2}\left(-\frac{2\pi l}{q_2}\right) = 0 \qquad l \neq 0$$

where q_1, q_2 are the step sizes of the quantizers, i.e.

$$E(x_1(t_1)\ x_2(t_2)) = E(y_1'(t_1)\ y_2'(t_2)) \qquad (2.3.12)$$

These conditions (eqn. (2.3.11)) imply that $n_1(t)$, $n_2(t)$ should be such that (i) they are zero mean (ii) they are independent of the noise signals and of each other (iii) their p.d.fs should be such that their characteristic functions have periodic zeros with period $2\pi/q_1$, $2\pi/q_2$ no zero lying at the origin.

It is shown [16] that the rectangular distribution

$$\begin{aligned} p(n) &= 1/q \qquad |n| \leq q/2 \\ &= 0 \qquad \text{elsewhere} \end{aligned} \qquad (2.3.13)$$

satisfies eqns. (2.3.11). The p.d.f's which are m-fold convolutions m = 1,2, ... of the rectangular distribution with itself also satisfy equation (2.3.11).

The general form of the characteristic function of this family of r.v.s. is

$$W_n(\alpha) = \prod_{i=1}^{\infty} \left[1 - \left(\frac{\alpha q}{2\pi i}\right)^2\right]^{m_i} \cdot F(\alpha) \qquad (2.3.14)$$

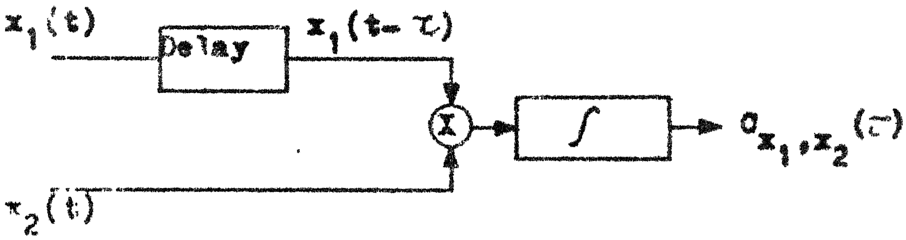
where m_i are nonzero positive integers, and $F(\alpha)$ is s.t. $F(\alpha) \leq 1$, $F(0) = 1$. For the corresponding auxiliary noise $n(t)$ to be zero mean, $F'(0) = 0$.

DEVICES FOR CORRELATION ESTIMATION : Correlation techniques are of practical importance in technology. An important use of these techniques is the measurement of target range and velocity by radar and sonar systems, and determination of the transfer functions of control systems. Correlation devices may be either analog or digital. The various categories are direct correlator, digital correlator [14], stieltjes correlator [14], modified digital (polarity coincidence) correlator [17], relay (modified stieltjes) correlator [18], modified relay correlator. Fig. (2.3.3) shows self explanatory schematics of these correlation devices.

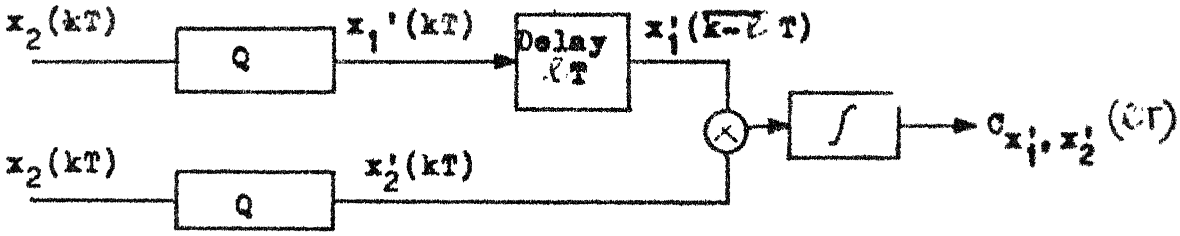
The Signum Function and its Approximations

The signum function is a step function defined as follows :

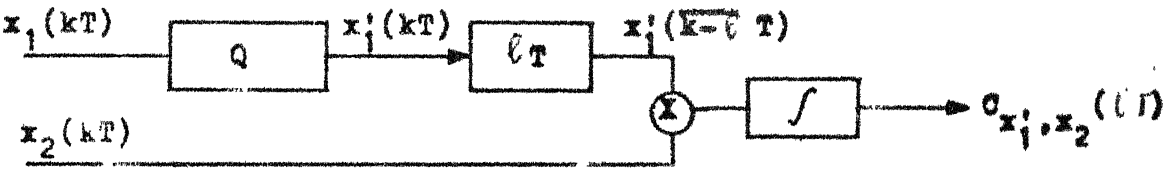
$$\operatorname{sgn}(x) \begin{cases} = -1 & x < 0 \\ = \ \ 0 & x = 0 \\ = \ \ 1 & x > 0 \end{cases} \qquad (2.3.15)$$



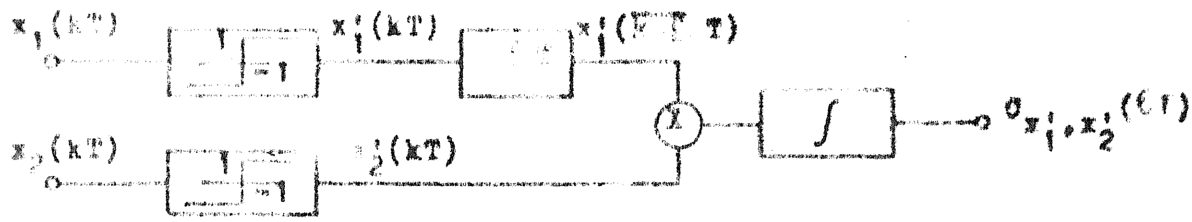
(a) Direct



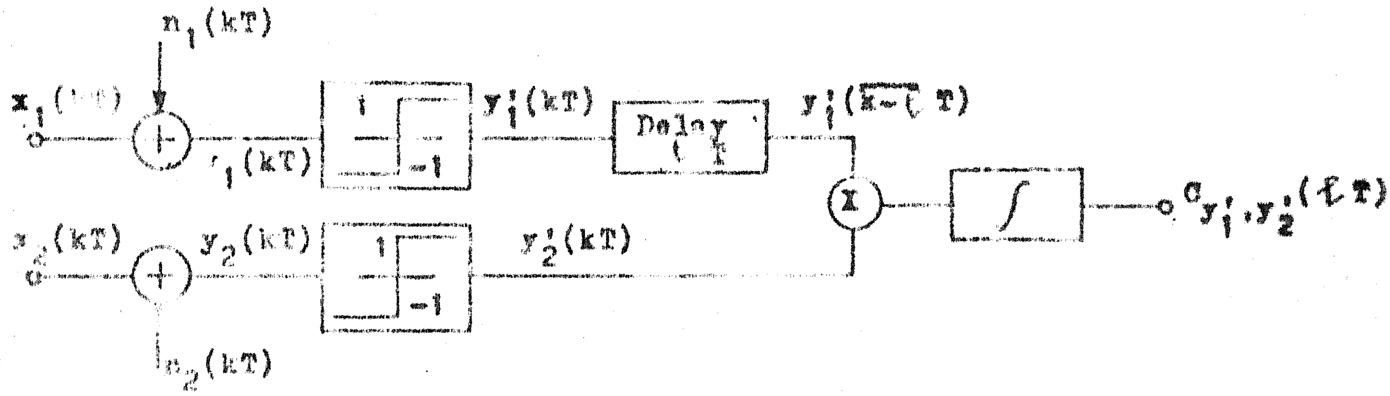
(b) Digital Correlator



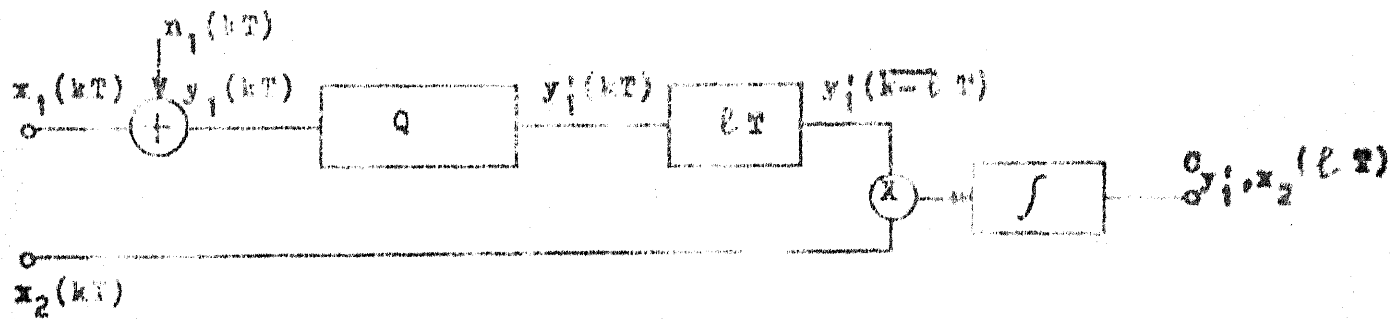
(c) Stieltjes Correlator



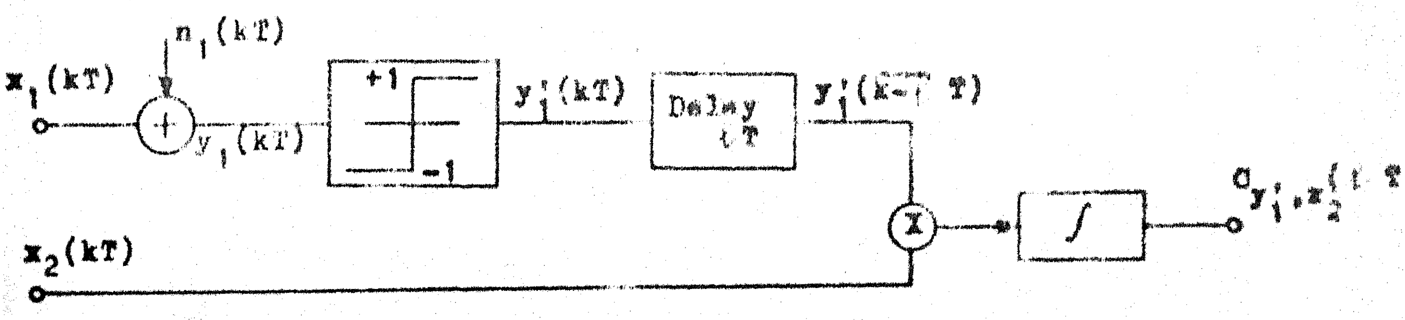
(d) Polarity Coincidence Correlator



(e) Modified Polarity Coincidence Correlator



(f) Relay (Modified Stieltjes) Correlator



(g) Modified Relay Correlator

Fig. 2.3.3

The unit function is defined thus

$$u(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases}$$

In terms of $u(x)$, therefore, $\mathrm{sgn}(x)$ is given by

$$\mathrm{sgn}(x) = 2u(x)-1 \qquad (2.3.16)$$

The unit step $u(x)$ may be represented as the limit of many continuous functions.

1. $$u(x) = \lim_{\alpha \to \infty} \left[\frac{1}{2} + \frac{1}{\pi} \tan^{-1}(\alpha x)\right] \qquad (2.3.17)$$

2. $$u(x) = \lim_{\alpha \to \infty} \frac{1}{2}\left[\mathrm{erf}(\alpha x) + 1\right] \qquad (2.3.18)$$

where $\mathrm{erf}(x) = \dfrac{2\alpha}{\sqrt{\pi}} \displaystyle\int_0^{x} \exp(-\alpha^2 y^2)\,dy$

3. $$u(x) = \lim_{\alpha \to \infty} \left[2^{(-e^{-\alpha x})} \right] \qquad (2.3.19)$$

4. $$u(x) = \lim_{\alpha \to \infty} \frac{1}{\pi} \int_{-\infty}^{\alpha x} \frac{\sin y}{y}\,dy \qquad (2.3.20)$$

These functions are illustrated in Fig. (2.3.1).

2.4 A Statistical Coarse-Sampled-Data Filter.

In a recent publication [7] a new technique for the hardware implementation of digital filters is introduced.

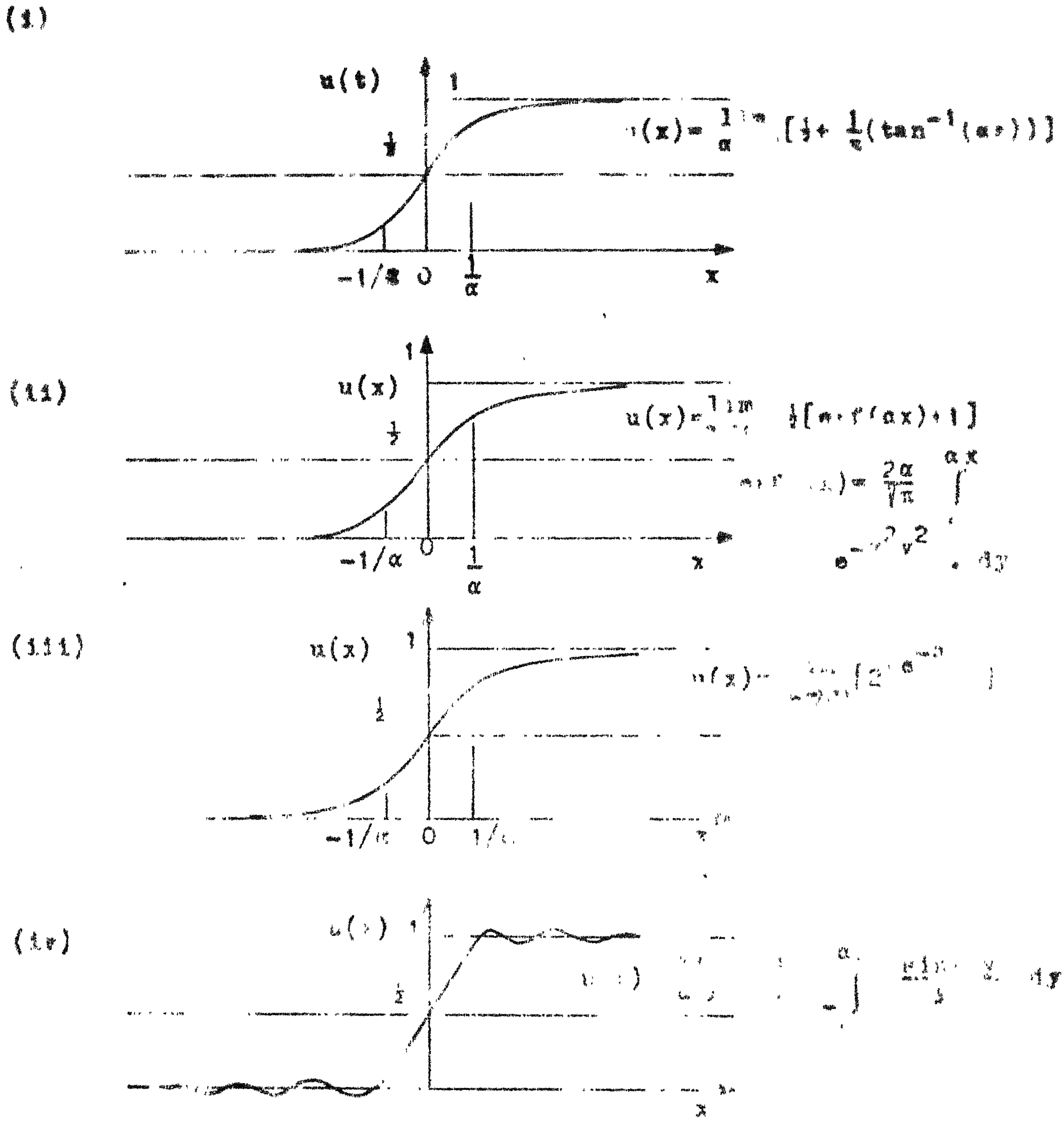


Fig. 2.3.4: Approximations of the Step Function

By use of this technique, the complexity of the hardware required is sought to be reduced. The proposed filter requires only one summer, one shift register of length equal to the order of the filter, either a simple RC low pass filter or a digital counter, and a generator of noise uniformly distributed in amplitude and independent between samples. The principle of operation is based on polarity coincidence correlation [8], Fig. (2.3.3d), and is similar to dithering employed to over-come the effects of quantization in video data for e.g. [9],[10]. It is used when low input sample rate are sufficient as this filter requires a higher sampling rates than the direct.

The proposed filter implementation technique is developed as follows:-

Let x_o be a random variable with probability density function $p(x_o)$.

Let $n(t)$ be a noise process, stationary, white, uniformly distributed in amplitude and independent between samples, lying in the range $-N \leq n \leq N$.

The range of x_o is included in that of $n(t)$, or at most the two ranges are equal. Assume that the range of x_o is equal to the range of $n(t)$ and let it be normalised so that

$$-1 \leq x_o \leq 1 \qquad (2.4.1)$$

Thus, the probability density of n is

$$p_n(\alpha) = \tfrac{1}{2} \quad -1 \le \alpha \le 1 \qquad (2.4.2)$$
$$= 0 \text{ elsewhere.}$$

Now consider the random variable z generated thus

$$z = \operatorname{sgn}(x_o + n(t)) \qquad (2.4.3)$$

The conditional density $p(z|x_o)$ is given by

$$p(z|x_o) = \delta(z-1).p(n(t) > -x_o) + \delta(z+1)p(n(t) < -x_o) \qquad (2.4.4)$$

$$= \delta(z-1)\ \tfrac{1}{2} \int_{-x_o}^{1} dn + \delta(z+1)\ \tfrac{1}{2} \int_{-1}^{-x_o} dn \qquad (2.4.5)$$

$$= \delta(z-1)\ \frac{1+x_o}{2} + \delta(z+1)\ \frac{1-x_o}{2} \qquad (2.4.6)$$

The conditional expectation is given by

$$E(z|x_o) = \int_z \alpha\, p(\alpha|x_o)\, d\alpha \qquad (2.4.7)$$

$$= \int_z \alpha\left[\delta(\alpha-1)\ \frac{1+x_o}{2} + \delta(\alpha+1)\ \frac{1-x_o}{2}\right] d\alpha \qquad (2.4.8)$$

$$= 1 . \frac{1+x_o}{2} + \frac{1-x_o}{2} . (-1) = x_o \qquad (2.4.9)$$

$$E(z) = E(x_o) \qquad (2.4.10)$$

An intuitive explanation of eqn. (2.4.10) may be given. Consider the case when x_o varies slowly in time

w.r.t. $n(t)$. The dithering signal $n(t)$ is added to x_o say N times in a period of Δ secs; each time the sum p is formed.

$$p = x_o + n(t)$$
$$\text{and} \quad z = \text{sgn}\ (p) \qquad (2.4.11)$$

We can see that if x_o be deterministic and $x_o=0$, because the probabilities of $n(t)$ being less than and greater than zero, are equal, therefore p shall be (+ve) or (-ve) with equal probability. This implies that z is +1 or -1 with equal probability or its mean shall indeed be zero which is the value of x_o in this case. If x_o be +ve (-ve), the tendency of z will be to be +1(-1) a larger number of times than -1(+1). Averaging z over a large number of $n(t)$ samples, will yield some estimate of x_o.

A digital filter may be given by the difference equation

$$y_k = \sum_{i=o}^{N} a_i u_{k-i} + \sum_{j=1}^{M} b_j y_{k-j} \qquad (2.4.12)$$

where u_k is the input and y_k the output. Consider the formulation

$$y_k = \text{sgn}\ [\ \sum_{i=o}^{N} a_i u_{k-i} + \sum_{j=1}^{M} b_j y_{k-j} + n_k] \qquad (2.4.13)$$

where n is a noise process with density as in (2.4.2), stationary , white, and independent between samples. Then by equation (2.4.10)

$$E(y_k) = \sum_{i=0}^{N} a_i u_{k-i} + \sum_{j=1}^{M} b_j \, E(y_{k-j}) \qquad (2.4.14)$$

where u_k is assumed deterministic. Equations (2.4.13) and (2.4.14) suggest a filter design, the output of which after averaging many samples yields an estimate of the filtered u_k. This is the quantized digital filter obtained from the linear or unquantized difference equation given by eqn. (2.4.12). The output y_k is followed by an averager, either an RC averager or a digital counter. The quantized filter obtained from the following difference equation

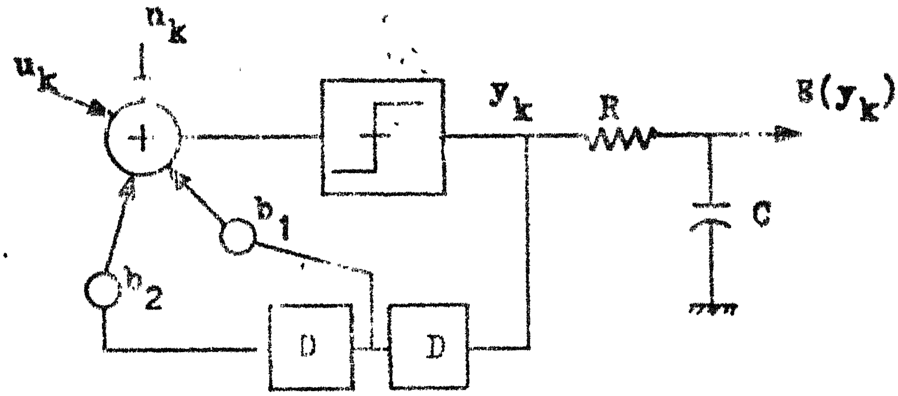
$$y_k = u_k + b_1 y_{k-1} + b_2 \, y_{k-2} \qquad (2.4.15)$$

is illustrated in Fig. (2.4.1).

Therefore the output of the quantized digital filter is an estimate of the true digital filter output. The output power spectral density is found by z-transforming the discrete ACF $R_{yy}(m)$ of the output.

$$H(z)^2 = \left| \frac{\sum_i a_i z^{-i}}{1 - \sum_j b_j z^{-j}} \right|^2 \qquad (2.4.16)$$

is the ratio of the output and input spectral densities.



2 bit shift register

Fig. 2.4.1: ORDER COARSE-SAMPLED-DATA FILTER

<u>Finite Averaging Errors</u>: **Since infinite time is not** available for integration, approximate **methods such as the** RC averager (a digital counter) must be used. **Let y be** scaled and shifted such that $0 \leq y \leq 1$ and $E(y) = p_o$. **Since** y is ergodic y = 1 for p_o fraction of the time and y=0 for $1-p_o$ fraction of the time. Since the dither is independent between samples, the auto-correlation of y is

$$\begin{aligned} R_{yy}(\ ) &= p_o^2 + (p_o - p_o^2)\left(1 - \frac{|\tau|}{T}\right) & |\tau| \leq T \\ &= p_o^2 & |\tau| > T \end{aligned} \qquad (2.4.17)$$

T being the sample and hold interval.

The variance σ_v^2 of the output of the averager is given by

$$\sigma_v^2 = (p_o - p_o^2)\left[1 - \frac{RC}{T} \cdot (1 - e^{-T/RC})\right] \qquad (2.4.18)$$

$$= (p_o - p_o^2) \cdot \frac{T}{2RC} \quad \text{as} \quad \frac{T}{2RC} \rightarrow 0 \qquad (2.4.19)$$

It shall be shown in Chapter 3 , that the output variance may be decreased by a change in the structure of the quantized filter, using a digital counter as the averager.

CHAPTER 3

DEVELOPMENT OF AN N-BIT FEEDBACK COARSE-SAMPLED-DATA FILTER

In this chapter the statistics of the output of first and second order linear digital filter are evaluated (Sec 3.1) Kirlin's coarse-sampled-data filter (Sec. 2.4) incorporating a digital counter averager in place of the RC averager is discussed in Sec. 3.2. The coarse-sampled-data filter is modified to yield the N-bit feedback coarse-sampled-data filter, discussed in Sec. 3.3, the output statistics of which can be made to approximate closely those of the output of the appropriate linear digital filter.

3.1 The Linear Digital Filter

A discrete time system is an algorithm that converts the input sequence u(nT) into another sequence (the output y(nT), the two sequences being related by the operator Ø

$$y(nT) = \emptyset\,[u(nT)] \tag{3.1.1}$$

A linear time invariant system is characterized by its impulse response h(n) [1,pp. 13-16]. The response to u(n) is given by the convolution relation

$$y(nT) = \sum_{m=-\infty}^{\infty} h(mT)u(\overline{n-m}T) \tag{3.1.2}$$

We shall consider causal, stable LTI systems. The input and output sequences to a LTI discrete system may be related by a constant coefficient Mth order difference equation of the form

$$y(nT)=\sum_{k=o}^{M} a_k u(\overline{n-k}T)+\sum_{k=1}^{M} b_k\, y(\overline{n-k}T) \quad n \geq 0 \qquad (3.1.3)$$

$\{a_i\}$, $\{b_i\}$ characterize the system and $b_M \neq 0$. The transfer function of the system represented by (3.1.3) is

$$H(z)=\frac{Y(z)}{U(z)}=\frac{\sum_{k=o}^{M} a_k z^{-k}}{1-\sum_{k=1}^{M} b_k\, z^{-k}} \qquad (3.1.4)$$

The frequency response of the system, $H(e^{j\omega})$ is given by

$$H(e^{j\omega})=H(z)\Big|_{z=e^{j\omega}}=\frac{\sum_{k=o}^{M} a(e^{-j\omega})^k}{1-\sum_{k=1}^{M} b_k(e^{-j\omega})^k} \qquad (3.1.5)$$

where ω is in radians.

We shall consider the specific cases of the first and second order difference equation i.e. when M=1 and M=2. The input to the system u(nT) may be either deterministic or

random. For each of the two cases the statistics of the output of the system shall be determined.

3.1.1 The First Order Filter:

The first order discrete time system is given by the difference equation

$$y(kT)=a_o u(kT)+a_1 u(\overline{k-1}T) + b_1 \; y(\overline{k-1}T) \qquad (3.1.6)$$

Consider the case when a_1=o for simplicity, $|b_1| < 1$ for stability, (3.1.6) reduces to

$$y(kT) = a_o \; u(kT)+b_1 \; y(\overline{k-1}T) \qquad (3.1.7)$$

Let the input to the system be u(nT) U(rT), U(rT) is the unit step function; by use of the recursive relationship of eqn. (3.1.7), it is easily shown that

$$y_{kT} = a_o \sum_{r=o}^{k} b_1^r \; u_{\overline{k-r}T} \qquad (3.1.8)$$

(a) <u>The Deterministic Input</u>: When u(kT) is deterministic, y(kT) is deterministic and from eqn. (3.1.8)

$$E(y_{kT}) = a_o \sum_{r=o}^{k} b_1^r \; E(u_{\overline{k-r}T}) \qquad (3.1.9)$$

$$= a_o \sum_{r=o}^{k} b_1^r \; u_{\overline{k-r}T} \qquad (3.1.10)$$

A time autocorrelation function $R_y(m)$ may be defined for the deterministic output y_{kT}, given by

$$R_y(m) = \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} y_{kT} . y_{\overline{k+m}T} \tag{3.1.11}$$

$$= \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} (a_o \sum_{r=o}^{k} b_1^r \, u_{\overline{k-r}T})(a_o \sum_{i=o}^{k+m} b_1^i \, u_{\overline{k+m-i}T}) \tag{3.1.12}$$

$$= \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} a_o^2 \, [\sum_{r=o}^{k} \sum_{i=o}^{k+m} b_1^{r+i} \, u_{\overline{k-r}T} u_{\overline{k+m-i}T}] \tag{3.1.13}$$

(b) <u>The random input</u>: Let u_{nT} be zero mean, white, stationary independent noise, specifically:

$$E(u_{nT}) = \eta_u = o \quad \text{all } n$$

$$E(u_{nT} \, u_{\overline{n+m}T}) = \sigma_u^2 \, \delta(m) \tag{3.1.14}$$

$$p(u_{nT}, u_{\overline{n+m}T}) = p(u_{\overline{n+m}T}) \; p(u_{nT}), \; m\neq 0$$

Then the expected value of the output y_{kT}, using the eqn. (3.1.8)

$$E(y_{kT}) = E[a_o \sum_{r=o}^{k} b_1^r \, u_{\overline{k-r}T}]$$

$$= a_o \sum_{r=o}^{k} b_1^r \, E(u_{\overline{k-r}T}) \tag{3.1.15a}$$

$$= 0 \tag{3.1.15b}$$

The ensemble autocorrelation of y(kT) is

$$R_y(k, k+m) = E(y_{kT}\, y_{\overline{k+m}T})$$

$$= a_o^2 \sum_{r=o}^{k} \sum_{i=o}^{k+m} b_1^{r+i}\, E[u_{\overline{k-r}T}\, u_{\overline{k+m-i}T}] \tag{3.1.16}$$

$$R_y(k, k+m) = a_o^2\, \sigma_u^2 \sum_{r=o}^{k} \sum_{i=o}^{k+m} \delta(m+r-i) b_1^{r+i} \tag{3.1.17}$$

The autocorrelation of y_{kT} may also be derived as follows:

$$E(y(kT)y(\overline{k+m}T)) = a_o^2\, E(u(kT)u(\overline{k+m}T)) + a_o b_1 [E(u(kT)y(\overline{k+m-1}T)) + E(y\overline{k-1}T\ ,\ u(\overline{k+m}T))] + b_1^2\, E(y(\overline{k-1}T)y(\overline{k+m-1}T)) \tag{3.1.18}$$

The cross correlation terms $R_{y,u}(k-1, k+m)$, $R_{u,y}(k, k+m-1)$ are first evaluated

(i) $R_{y,u}(k-1, k+m) = E[y_{k-1}\, u_{k+m}]$

$$m > -1 \quad R_{y,u}(k-1, k+m) = 0 \tag{3.1.19}$$

This is because $y(\overline{k-1}T)$ is a function only of $u(\overline{k-1}T)$ and all the previous values of u, and is independent of the future values of the input.

$m = -1$

$$R_{y,u}(k-1, k-1) = E[y_{k-1}\, u_{k-1}]$$

$$= E[(a_o u_{k-1} + b_1\, y_{k-2})\, u_{k-1}]$$

$$= a_o\, \sigma_u^2 + 0 \tag{3.1.20}$$

$m < -1$

$$R_{y,u}(k-1,k+m) = E[y_{k-1}\ u_{k+m}]$$
$$= E[(a_o u_{k-1}+b_1 y_{k-2})u_{k+m}]$$
$$= 0 + a_o\ \sigma_u^2\ b_1^{|m+1|} \qquad (3.1.21)$$

(ii) $R_{u,y}(k,k+m-1) = E[u_k\ y_{k+m-1}]$

$m < 1$

$$E[u_k y_{k+m-1}] = 0 \qquad (3.1.22)$$

$m = 1$

$$E[u_k\ y_{k+m-1}] = E[u_k y_k]$$
$$= a_o\ \sigma_u^2 \qquad (3.1.23)$$

$m > 1$

$$E[u_k y_{k+m-1}] = a_o\ \sigma_u^2\ b_1^{m-1} \qquad (3.1.24)$$

Therefore,

$$R_y(k,k+m) = a_o^2\ (1+b_1^2+(b_1^2)^2 + \ldots + (b_1^2)^k) \quad m=0$$
$$= a_o^2\ b_1^m\ [(1+b_1^2+b_1^4+ \ldots + b_1^{2k}).\sigma_u^2]\ m > 0$$
$$= a_o^2\ \sigma_u^2\ [1+b_1^2+b_1^4 \ldots + b_1^{2(k+m)}]\ b_1^{m} \quad m < 0$$

$$R_y(k,k+m) = \frac{a_o^2\ 1-b_1^{2k+2}}{1-b_1^2} \cdot \sigma_u^2 \quad m = 0$$
$$= a_o^2\ \sigma_u^2\ b_1^m\ \frac{1-b_1^{2k+2}}{1-b_1^2} \qquad m>0 \qquad (3.1.25)$$
$$= a_o^2\ \sigma_u^2\ b_1^m\ (1-b_1^{2(k+m)+2})(1-b_1^2)^{-1} \quad m< 0$$

The input to the system is u_{nT} U(jT), U(jT) being the unit step function. The process is necessarily nonstationary, regardless of the nature of u_{nT}. Hence the output of the system y_{nT} is nonstationary, which fact is ratified by eqn. (3.1.25). When the system reaches steady state i.e. with k very large, and $\because |b_1| < 1$, eqn. (3.1.25) tends to

$$R_y(m) = \frac{a_o^2\ \sigma_u^2}{1-b_1^2} \qquad m = 0$$

$$= \frac{a_o^2\ \sigma_u^2}{1-b_1^2}\ \ b_1^{\,m} \qquad |m| \geqslant 1 \tag{3.1.26}$$

In steady state the power spectrum of the output is

$$S_y(\omega) = \sum_{m=-\infty}^{\infty} R_y(m)\ e^{-j\omega m} \tag{3.1.27}$$

$$= \frac{a_o^2\ \sigma_u^2}{1-b_1^2} + \frac{a_o^2\ \sigma_u^2}{1-b_1^2} \sum_{m=1}^{\infty} [b_1^m\ e^{-j\omega m} + b_1^m\ e^{j\omega m}]$$

$$S_y(\omega) = \frac{a_o^2\ \sigma_u^2}{1-b_1^2}\ [\ 1+2 \sum_{m=1}^{\infty} b_1^m\ \cos\omega m\ ] \tag{3.1.\ \ }$$

The value for $S_y(\omega)$ obtained using the relation $S_y(\omega) = \sigma_u^2\ H(z)\ H(z^{-1})\Big|_{z=e^{j\omega}}$ for a two sided input and that obtained in eqn. (3.1.28) are equal. In steady state, the usual expression for $S_y(\omega)$ may be used.

3.1.2 The Second Order Filter:

The second order linear system is defined by the difference equation

$$y(kT) = a_o\ u(kT) + b_1\ y(\overline{k-1}T) + b_2\ y(\overline{k-2}T) \qquad (3.1.29)$$

The transfer function of the system is given by

$$H(z) = \frac{a_o}{1-b_1\ z^{-1} - b_2 z^{-2}} \qquad (3.1.30)$$

The inverse-z transform of H(z) gives the impulse response h(nT) of the system. y(kT) is given by the convolution integral

$$y(kT) = \sum_{m=o}^{\infty} u(mT)\ h(\overline{k-m}T) \qquad (3.1.31)$$

H(z) is given by eqn. (3.1.30)

$$H(z) = a_o\ (1-p_1 z^{-1})^{-1}\ .\ (1-p_2\ z^{-1})^{-1}$$

$$\text{where } p_1 = \frac{b_1+(b_1^2+4b_2)^{1/2}}{2},\ p_2 = \frac{b_1-(b_1^2+4b_2)^{1/2}}{2} \qquad (3.1.32)$$

$$H(z) = a_o(1+p_1 z^{-1}+p_1^2 z^{-2}+p_1^3 z^{-3}+\ldots)(1+p_2 z^{-1}+p_2^2 z^{-2}+p_2^3 z^{-3}+\ldots)$$

$$= a_o \left\{ 1+ \sum_{n=1}^{\infty} z^{-n} \left[\sum_{i=o}^{n} p_1^i\ p_2^{n-i} \right] \right\} \qquad (3.1.33)$$

$$\text{Hence } h(nT) = 0 \qquad n < 0$$

$$= a_o \sum_{i=o}^{n} p_1^i\ p_2^{n-i} \qquad n \geqslant 0 \qquad (3.1.34)$$

(a) The Deterministic Input: The RHS of eqn. (3.1.31) gives both the output of the system and its expected value, when $u(kT)$ is deterministic. The time autocorrelation is given by $R_y(m)$

$$R_y(m) = \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=0}^{K} y(kT)\, y(\overline{k+m}T)$$

$$= \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=0}^{K} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} h(iT)u(\overline{k-i}T)h(jT)u(\overline{k+m-j}T)$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} h(iT)h(jT) \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=0}^{\infty} u(\overline{k-i}T)u(\overline{k+m-j}T) \qquad (3.1.35)$$

Now if the time autocorrelation function of $u(kT)$ is

$$R_u(m) = \lim_{k\to\infty} \frac{1}{K+1} \sum_{k=0}^{\infty} u(kT)\, u(\overline{k+m}T)$$

substituting this relation in (3.1.35) we obtain

$$R_y(m) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} h(iT)h(jT).R_u(m-j+i) \qquad (3.1.36)$$

(b) The Random Input: to the system is $u(nT)\,U(jT)$, where $u(nT)$ is defined by eqn. (3.1.14) and $U(jT)$ is the unit step function. From eqn. (3.1.31) it is deduced that

$$E(y(kT)) = E\left[\sum_{m=0}^{\infty} u(mT)h((\overline{k-m}T)\right] \qquad (3.1.37)$$

$$= \sum_{m=0}^{\infty} E[u(mT)]\, h(\overline{k-m}T) = 0 \qquad (3.1.38)$$

The autocorrelation of y(kT) is defined as

$$R_y(k,k+m) = E[\ y(kT)y(\overline{k+m}T)] \qquad (3.1.39)$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} h(\overline{k-n}T)h(\overline{k+m-j}T)\sigma_u^2\ \delta(n-j) \qquad (3.1.40)$$

$$= \sigma_u^2 \sum_{n=0}^{\infty} h(\overline{k-n}T)\ h(\overline{k+m-n}T) \qquad (3.1.41)$$

Obtaining the expression for the non stationary $R_y(k,k+m)$ by using eqns. (3.1.41) and (3.1.34) setting $k\to\infty$ for steady state conditions and obtaining the DFT of the stationary $R_y(m)$ is unnecessary as the relation

$$S_y(\omega)= S_y(z)\Big|_{z=e^{j\omega}} = \sigma_u^2\ H(z)\ H(z^{-1})\Big|_{z=e^{j\omega}} \qquad (3.1.42)$$

to obtain the power spectrum for a two sided input yields the same result. Therefore

$$S_y(\omega) = \frac{\sigma_u^2\ a_o^2}{(1-b_1 e^{-j\omega} - b_2 e^{-j2\omega})(1-b_1\ e^{j\omega} - b_2\ e^{j2\omega})} \qquad (3.1.43)$$

$$= \frac{\sigma_u^2 \cdot a_o^2}{1+b_1^2+b_2^2-2(b_1+b_1 b_2)\cos\omega-2b_2\ \cos\ 2\omega} \qquad (3.1.44)$$

The autocorrelation function is given by the inverse-z transform of $S_y(z)$

$$\checkmark\quad R_y(mT) = \frac{1}{2\pi j} \int S_y(z)\ .\ z^{m-1}\ dz \qquad (3.1.45)$$

For stable H(z), all its poles lie within the unit circle $z = e^{j2\pi fT}$ ... (z) exists on the unit circle and $R_y(mT)$ is given by

$$R_y(mT) = T\int_{-\frac{1}{2T}}^{\frac{1}{2T}} \left|H(z=e^{j2\pi fT})\right|^2 . \exp\,(j\pi fTk).\, df \qquad (3.1.46)$$

The second order difference equation may be simplified by setting $b_1=0$, in which case the recursive relationship is

$$y(kT) = a_o\, u(kT) + b_2 y(\overline{k-2T}) \qquad (3.1.47)$$

The transfer function of the system is

$$H(z) = \frac{a_o}{1-b_2 z^{-2}}$$

$$= \frac{a_o}{(1-p_1 z^{-1})(1-p_2 z^{-1})} \qquad (3.1.48)$$

$$p_1 = \sqrt{b_2}\ ,\ p_2 = -\sqrt{b_2}$$

The steady state power spectrum of the output is

$$S_y(\omega) = S_y(z)\Big|_{z=e^{j\omega}} = \sigma_u^2\, H(z)\, H(z^{-1})\Big|_{z=e^{j\omega}}$$

$$S_y(\omega) = \frac{\sigma_u^2\, a_o^2}{1+b_2^2 - 2b_2\cos 2\omega} \qquad (3.1.49)$$

Using eqn. (3.1.47), y(kT) may be written in terms of the input values (u(iT), i=0,1,... [illegible] **n+1 or k=2n, as follows:**

$$y(kT) = a_o u(kT) + a_o b_2 u(\overline{k-2}T) + \ldots + a_o b_2^r u(\overline{k-2r}T)$$

$$+ \ldots + b_2^{2n}\, u(o) \quad \text{for even } k \qquad (3.1.50a)$$

$$= a_o u(kT) + a_o b_2 u(\overline{k-2}T) + \ldots + a_o b_2^r\, u(\overline{k-2r}T)$$

$$+ \ldots + b_2^{2n}\, u(1) \quad \text{for odd } k \qquad (3.1.50b)$$

Thus y(kT), k is odd (even), is a weighted sum of the odd (even) input values.

Evaluation of the Autocorrelation function $R_y(k,k+m)$ of the output y(kT) requires prior knowledge of the cross-correlation functions viz. $R_{uy}(k,k+m-2)$ and $R_{yu}(k-2,k+m)$. Using eqn. (3.1.50)

$$R_{u,y}(k,k+m-2) = E[u(kT)y(\overline{k+m-2}T)]$$

$$= 0 \qquad m < 2, m = \text{all odd (+ve) integers}$$

$$= a_o \sigma_u^2 (b_2)^{\frac{m-1}{2}} \qquad m = \text{all even(+ve) integers.}$$

(3.1.51)

$$R_{y,u}(k-2,k+m) = E[y(\overline{k-2}T)u(\overline{k+m}T))$$

$$= 0 \qquad m > -2, m = \text{odd (-ve) integers}$$

$$= a_o \sigma_u^2 (b_2)^{\frac{|m|}{2}-1} \qquad m = \text{all even(-ve) integers}$$

(3.1.52)

Now

$$R_y(k,k+m)=a_o^2\ E[u(kT)u(\overline{k+m}T)]+a_ob_2\ E[y(\overline{k-2T})u(\overline{k+m}T)]$$

$$+\ a_ob_2\ E[u(kT)y(\overline{k+m-2}T)+b_2^2\ E[y(\overline{k-2}T)y(\overline{k+m-2T})]$$

(3.1.53)

Substituting (3.1.51) and (3.1.52) in the recursive eqn. (3.1.53), for k=2n+1 or k=2n and k+m=2r or k+m=2r+1

$$R_y(k,k+m)=a_o^2\ \sigma_u^2\ (1+b_2^2+b_2^4+\ldots+(b_2^2)^n),\quad m=0$$

$$=a_o^2\ \sigma_u^2\ [\frac{1-b_2^{2n+2}}{1-b_2^2}]\qquad (3.1.54a)$$

$$=a_ob_2.a_o\sigma_u^2b_2^{m/2-1}[1+b_2^2+b_2^4+\ldots+(b_2^2)^n]\quad m=2,4,6,\ldots$$

$$=a_o^2\ \sigma_u^2\ b_2^{m/2}\ \frac{1-b_2^{2n+2}}{1-b_2^2}\qquad (3.1.54b)$$

$$=a_o^2\ b_2\ \sigma_u^2.b_2^{|m|/2\ -1}[1+b_2^2+b_2^4+\ \ldots(b_2^2)^r]$$

$$m=-2,-4,-6,\ldots$$

$$=a_o^2\sigma_u^2.b_2^{|m|/2}\ [\frac{1-b_2^{2r+2}}{1-b_2^2}]$$

(3.1.54c)

$$=0\quad |m|\ =\ 1,3,5,\ \ldots\qquad (3.1.54d)$$

In steady state conditions when k is large, the Autocorrelation is stationary i.e.

$$R_y(m) = \lim_{k \to \infty} R_y(k, k+m)$$

$$= \frac{a_o^2 \sigma_u^2}{1-b_2^2} \qquad m = 0$$

$$= \frac{a_o^2 \sigma_u^2}{1-b_2^2} \; b_2^{|m|/2} \qquad |m| = 2,4,6,8,\ldots \tag{3.1.5}$$

$$= 0 \qquad \text{elsewhere}$$

3.2 The Coarse-Sampled-Data Filter.

In (Chap. 2.4), the basic coarse-sampled-data filter using an RC averager, was studied. Here we study the coarse-sampled-data filter using a digital counter averager for comparison with the linear digital filter and for the subsequent comparison of its output with the output of its modification.

3.2.1 The First Order Coarse-sampled-data filter

The difference equation characterising the first order linear digital filter is

$$y(lT) = a_o\, u(lT) + b_1 y(\overline{l-1}T) \tag{3.2.1}$$

The filter is replaced by the coarse-sampled-data filter which serves to reduce the complexity of the hardware realisation of the filter. The relation between the input $u(lT)$ and the output $\hat{y}(lT)$ of the coarse-sampled-data filter is given by the equations

$$z(Nl+k) = \mathrm{sgn}[a_o u(lT)+n(Nl+k)+b_1 z(Nl+k-1)] \tag{3.2.2a}$$

$$\hat{y}(lT) = \frac{1}{N} \sum_{k=1}^{N} z(Nl+k) \tag{3.2.2b}$$

A schematic diagram of the filter is shown in Fig. (3.2.1). Referring to the figure, let $u(lT)$, the input to the coarse-sampled-data filter, be defined as in eqn. (3.1.14). The input and output sampling periods of the coarse-sampled-data filter are T secs. T should be chosen such that it is commensurate with the Nyquist rate of the input. The input should also be effectively constant over the sampling period T. The output of the ZOH is

$$u(Nl+k) \triangleq u(\overline{Nl+k}T') = u(lT) \quad \text{for} \quad 1 \leqslant k \leqslant N \tag{3.2.3}$$

The dithering noise $n(Nl+k)$ added is such that it is uniformly distributed in the range $(-1,1)$, independent between samples, and white. It is normalized, so that

$$\begin{aligned} &[a_o\, u(lT)]_{max} + |b_1| \leqslant 1 \\ \text{and } &[a_o\, u(lT)]_{min} - |b_1| \geqslant -1 \end{aligned} \tag{3.2.4}$$

The output of the comparator or sign detector is $+1$ (-1) as its input, viz. $a_o u(lT)+n(Nl+K)+b_1 z(Nl+K-1)$, is greater than or (less than) zero. The delay of the feed back register is T', which is the period of the dither

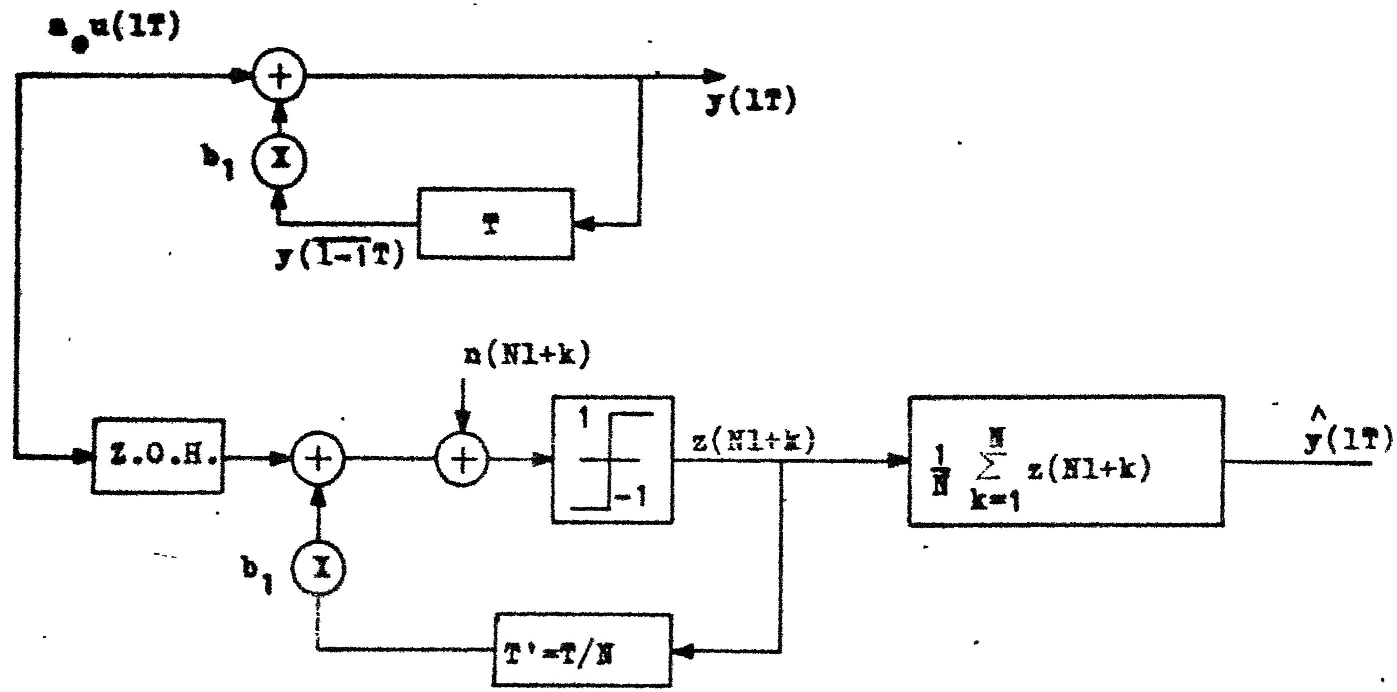


Fig. 3.2.1 Schematic of the first order linear digital filter and first order coarse-sampled-data filter.

$n(Nl+k)$. The averager is a digital counter that is reset to zero every lT secs, $l=0,1,\ldots$ It forms the weighted sum $\frac{1}{N}\sum_{k=1}^{N} z(Nl+k)$, upcounting (down counting) by one as $z(Nl+K)$ is $+1(-1)$, effectively averaging out the effect of the added dithering noise $n(Nl+k)$. The initial value of the delay line, $z(-N+N)$, (in this case it is a single element corresponding to T') is ± 1 with equal probability.

(3.2.2) gives the input-output relationship of the coarse-sampled-data filter. The statistics of the output $y(lT)$ are evaluated for $l=0,1,2,\ldots$ for the case when the input $u(lT)$ is deterministic and random.

(a) <u>The Deterministic Input</u>: The range of the deterministic input to the system $u(nT)$ is such that eqn. (3.2.4) holds. The output $y(lT)$ is given by eqn. (3.2.2).

(i) The probability of $z(Nl+k)$: The single element $z(Nl+k-1)$ is initialised at $l=o$ and $k=1$, so that the probability of the initial value $z(o)=z(N-N)$ being $+1$ or -1 is half i.e.

$$p_{z_o}(1) = \tfrac{1}{2} = p_{z_o}(-1) \qquad (3.2.5)$$

From eqn. (3.2.2a), for $l > 0$, $k=1,2,\ldots,N$,

$$z(Nl+k)=1 \Rightarrow a_o u(lT)+n(Nl+k)+b_1 z(Nl+k-1) > 0$$

$$\Rightarrow n(Nl+k) > -a_o u(lT)-b_1 z(Nl+k-1) \qquad (3.2.6)$$

The conditional probability $p_{z_{Nl+k} \mid z_{Nl+k-1}}(1 \mid z(Nl+k-1))$ is given by

$$p_{z_{Nl+k} \mid z_{Nl+k-1}}(1 \mid z(Nl+k-1)) = p[n(Nl+k) > -a_o u(lT) - b_1 z(Nl+k-1)] \tag{3.2.7}$$

$$= \int_{-a_o u(lT) - b_1 z(Nl+k-1)}^{1} p(n).dn \tag{3.2.8}$$

$$= \frac{1 + a_o u(lT) + b_1 z(Nl+k-1)}{2} \tag{3.2.9}$$

Therefore

$$p_{z_{Nl+k} \mid z_{Nl+k-1}}(-1 \mid z(Nl+k-1)) = \frac{1 - a_o u(lT) - b_1 z(Nl+k-1)}{2} \tag{3.2.10}$$

Since $z(Nl+k-1) = \pm 1$, the four conditional densities may be evaluated from eqns. (3.2.9), (3.2.10).

$$p_{z_{Nl+k}}(1) = \sum_{z_{Nl+k-1}} p_{z_{Nl+k} \mid z_{Nl+k-1}}(1 \mid z(Nl+k-1))[p_{z_{Nl+k-1}}(z(Nl+k-1))] \tag{3.2.11}$$

$$= \frac{1 + a_o u(lT) + b_1}{2} \, p_{z_{Nl+k-1}}(1) + \frac{1 + a_o u(lT) - b_1}{2} \, . \, [p_{z_{Nl+k-1}}(-1)]$$

$$p_{z_{Nl+k}}(1) = \frac{1 + a_o u(lT) - b_1}{2} + b_1 \, p_{z_{Nl+k-1}}(1) \tag{3.2.12a}$$

From the recursive relationship of eqn. (3.2.12a) and from the initialisation $z(-N+N)= \pm 1$ with equal probability, the probability of $z(Nl+k)$ may be determined. Similarly

$$p_{z_{Nl+k}}(-1) = \frac{1-a_o\, u(lT)-b_1}{2} + b_1\, p_{z_{Nl+k-1}}(-1) \qquad (3.2.12b)$$

(ii) The Expected value of $\hat{y}(lT)$:

$\hat{y}(lT)$ is a function of both the input $u(lT)$ and the added dither $n(Nl+k)$ as in eqn. (3.2.2).

$$\hat{y}(lT) = \frac{1}{N} \sum_{k=1}^{N} z(Nl+k)$$

$$E(\hat{y}\,(lT)) = \frac{1}{N} \sum_{k=1}^{N} E[z(Nl+k)] \qquad (3.2.13)$$

From eqn. (3.2.2a),

$$E(z(Nl+k))=E[\mathrm{sgn}(a_o u(lT)+n(Nl+k)+b_1 z(Nl+k-1)] \qquad (3.2.14)$$

$$= \sum_{z_{Nl+k-1}} \left[\int_n [\mathrm{sgn}(a_o u(lT)+n(Nl+k)+b_1 z(Nl+k-1))] . \right.$$

$$\left. p_n(n)\, dn \right] \; . \; p(z(Nl+k-1)) \qquad (3.2.15)$$

Invoking the results shown in (Chap. 2.3), (3.2.14) reduces to

$$E[z(Nl+k)]=E[a_o u(lT)+b_1\; z(Nl+k-1)]$$

$$=a_o\; u(lT)+\; b_1\; E[z(Nl+k-1)] \qquad (3.2.16)$$

$$=a_o\; u(lT)[1+b_1+b_1^2+\; \ldots + \; b_1^{k-1}\;] +$$

$$b_1^K[a_o u(\overline{l-1}T)(1+b_1+b_1^2+\ldots+b_1^{N-1})+a_o u(\overline{l-2}T).b_1^N(1+b_1+b_1^2+\ldots b_1^{N-1})$$

$$+\ldots+a_o u(\overline{l-r}T)\ b_1^{N(r-1)}\ (1+b_1+b_1^2+\ldots b_1^{N-1})$$

$$+\ a_o\ u(oT)\ b_1^{N(l-1)}\ (1+b_1+b_1^2+\ \ldots\ b_1^{N-1}\ )]$$

$$= a_o\ u(lT).\ \frac{1-b_1^k}{1-b_1} + a_o\ b_1^k\ \frac{(1-b_1^N)}{1-b_1}\ .\ \sum_{r=1}^{l} b_1^{N(r-1)}\ .u(\overline{l-r}T) \tag{3.2.17}$$

Substituting (3.2.17) in (3.2.13) we obtain

$$E(\hat{y}(lT)) = \frac{1}{N} \sum_{k=1}^{N} E(z(Nl+k))$$

$$= \frac{1}{N}\{\frac{N\ a_o u(lT)}{1-b_1} - \frac{a_o u(lT)}{1-b_1} \sum_{k=1}^{N} b_1^k + a_o \frac{(1-b_1^N)}{(1-b_1)}$$

$$[\sum_{r=1}^{l} b_1^{N(r-1)}\ u(\overline{l-r}T)].[\sum_{k=1}^{N} b_1^k\ ]\}$$

$$= \frac{a_o u(lT)}{1-b_1} - \frac{b_1 a_o u(lT)\ (1-b_1^N)}{N\ (1-b_1)^2} + \frac{a_o b_1}{N}[$$

$$(\frac{1-b_1^N}{1-b_1})^2\ .\ \sum_{r=1}^{l} b_1^{N(r-1)}\ u(\overline{l-r}T)\ ] \tag{3.2.18}$$

For effective averaging, the length N of the counter must be very large. Therefore with $N\to\infty$, from (3.2.18) we obtain

$$E(\hat{y}(lT)) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} E(z(Nl+k))$$

$$= \frac{a_o u(lT)}{1-b_1} \tag{3.2.19}$$

By comparing eq. (3.2.19) and eq. (3.1.9), it is seen that the expected value of the output of the coarse-sampled data filter does not approximate the output of the filter it is required to replace.

(ii) The Time Autocorrelation of $\hat{y}(lT)$:

It is required to check the matching of the time autocorrelation functions $R_y(m)$, $R_{\hat{y}}(m)$ of the outputs of the linear digital and the coarse-sampled-data filters respectively. Let $R_{\hat{y}}(m)$ be defined as follows:

$$R_{\hat{y}}(m) = \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=0}^{K} \hat{y}(kT)\, \hat{y}(\overline{k+m}T) \tag{3.2.20}$$

$$= \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=0}^{K} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} z(Nk+i)\, z(\overline{Nk+m}+j) \tag{3.2.21}$$

$$= \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=0}^{K} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathrm{Sgn}(a_0 u(kT)+ n(Nk+i)+b_1 z(Nk+i-1)).\mathrm{Sgn}(a_0\, u(\overline{k+m}T)+n(\overline{Nk+m+j}) +b_1 z(\overline{Nk+m+}j-1)) \tag{3.2.22}$$

Now since $\hat{y}(k)$ is a function of the added dither, the expected value of $R_{\hat{y}}(m)$ is shown to be

$$E(R_{\hat{y}}(m)) = \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=0}^{K} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} E(z(Nk+i).z(\overline{Nk+m}+j)) \tag{3.2.23}$$

When m=o, for a finite N, $E[R_y(m)]$ reduces to

$$E(R_{\hat{y}}(o)) = \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} \frac{1}{N^2} \Big[\sum_{i=1}^{N} E(z^2(Nk+i))$$

$$+ \sum_{\substack{i=1 \\ i\neq j}}^{N} \sum_{j=1}^{N} E(z(Nk+i)z(Nk+j)\Big]$$

$$= \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} \frac{1}{N^2} \Big[N + \sum_{\substack{i=1 \\ i\neq j}}^{N} \sum_{j=1}^{N} E(z(Nk+i)$$

$$z(NK+j))\Big] \tag{3.2.24}$$

Using eqn. (3.2.17) and the results derived in (Chap. 2.3), (3.2.24) reduces to the recursive equation:

$$E(R_{\hat{y}}(o)) = \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} \frac{1}{N^2} \Big[(N^2-N)a_o^2\ u^2(kT) + 2a_o b_1 (N-1)$$

$$\left\{\frac{Na_o u(kT)}{1-b_1} - a_o u(kT).\ \frac{1-b_1^N}{1-b_1^2} + a_o\left(\frac{1-b_1^N}{1-b_1}\right)^2.\ \sum_{r=1}^{k} b_1^{N(r-1)}\ u(\overline{k-r}T)\right\}$$

$$+ b_1^2 \sum_{\substack{i=1 \\ i\neq j}}^{N} \sum_{j=1}^{N} E[z(\quad\quad z(Nk+j-1)]] \tag{3.2.25}$$

In the limiting case, as $N\to\infty$, $\quad_y\quad$, k)) tends to

$$\lim_{N\to\infty} E(R_{\hat{y}}(o)) = \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} \Big[a_o^2 u^2(kT) + 2a_o^2 b_1 \frac{u(kT)}{1-b_1} +$$

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{\substack{i=1 \\ i\neq j}}^{N} \sum_{j=1}^{N} E[z(Nk+i-1)z(\quad\ j-1)]] \tag{3.2.26}$$

Comparing Eqn. (3.2.26) with Eqn. (3.1.13) $\lim_{m=o, N\to\infty} E(R_{\hat{y}}(k,k))$ as given by (3.2.26) is seen to be greater than $R_{\hat{y}}(k,k)$ evaluated for the linear digital filter. Similarly for m≠o, from eqn. (3.2.23) and using (3.2.17)

$$E(R_{\hat{y}}(m)) = \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} E(z(Nk+i)z(N\overline{k+m}+j))$$

$$= \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} \frac{1}{N^2} [N^2 a_o^2 u(kT)u(\overline{k+m}T) + N\, a_o b_1 u(\overline{k+m}T).[\sum_{i=1}^{N} E(z(Nk+i-1))] + N\, a_o b_1 u(kT) \sum_{j=1}^{N} E(z(N\overline{k+m}+j-1))$$

$$+ b_1^2 \sum_{i=1}^{N} \sum_{j=1}^{N} E(z(Nk+i-1)z(N\overline{k+m}+j-1))] \qquad (3.2.27)$$

$$= \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} \frac{1}{N^2} [N^2 a_o^2 u(kT)u(\overline{k+m}T) + N\, a_o b_1 u(\overline{k+m}T).$$

$$\left\{E(z(N\overline{k-1}+N)) + \sum_{i=2}^{N} E(z(Nk+i-1))\right\} + Na_o b_1 u(kT).$$

$$\left\{E(z(Nk+m-1+N)) + \sum_{r=2}^{N} E(z(N\overline{k+m}+j-1))\right\} + b_1^2 \sum_{i=1}^{N} \sum_{r=1}^{N}$$

$$E(z(Nk+i-1)z(Nk+m+j-1))] \qquad (3.2.28)$$

$$= \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=0}^{K} \frac{1}{N^2} [N^2 a_0^2 u(kT)u(\overline{k+m}T) + Na_0 b_1 u(\overline{k+m}T) \cdot$$

$$[\{ a_0 u(\overline{k-1}T) \cdot \frac{1-b_1^N}{1-b_1} + a_0 b_1^N \cdot \frac{1-b_1^N}{1-b_1} \cdot \sum_{r=1}^{k-1} b_1^{N(r-1)} u(\overline{k-1-r}T)\}$$

$$+ \{(N-1)a_0 \frac{u(kT)}{1-b_1} - \frac{a_0 u(kT)}{1-b_1} \cdot \frac{b_1 - b_1^N}{1-b_1} + a_0 \frac{1-b_1^N}{1-b_1} \cdot \frac{b_1-b_1^N}{1-b_1} \cdot$$

$$\sum_{r=1}^{k} b_1^{N(r-1)} u(\overline{k-r}T)\}] + Na_0 b_1 u(kT) [\{ a_0 u(\overline{k+m-1}T) \cdot \frac{1-b_1^N}{1-b_1}$$

$$+ a_0 b_1^N \frac{1-b_1^N}{1-b_1} \sum_{r=1}^{k+m-1} b_1^{N(r-1)} u(\overline{k+m-1-r}T)\} + \{(N-1) \frac{a_0 u(\overline{k+m}T)}{1-b_1}$$

$$- \frac{a_0 u(\overline{k+m}T)}{1-b_1} \cdot \frac{b_1-b_1^N}{1-b_1} + a_0 \frac{1-b_1^N}{1-b_1} \cdot \frac{b_1-b_1^N}{1-b_1} \sum_{r=1}^{k+m} b_1^{N(r-1)} \cdot$$

$$u(\overline{k+m-r}T)\}] + b_1^2 \sum_{i=1}^{N} \sum_{j=1}^{N} E(z(Nk+i-1)z(\overline{Nk+m+}j-1))] \tag{3.2.29}$$

In the limit, as $N\to\infty$, $E\,[R_{\hat{y}}(m)]_{m\neq 0}$ tends to

$$\lim_{N\to\infty} E(R_{\hat{y}}(m))_{m\neq 0} = \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=0}^{K} [a_0^2 u(kT)u(\overline{k+m}T) + \frac{2a_0^2 b_1}{1-b_1} \cdot$$

$$u(kT)u(\overline{k+m}T) + b_1^2 \lim_{N\to\infty} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} E(z(Nk+i-1)$$

$$z(\overline{Nk+m+j-1}))] \tag{3.2.30}$$

Eqn. (3.2.30) is a recursive equation; comparing it with (3.1.13) for $m \neq o$, we see it yields extraneous terms apart from those encountered in eqn. (3.1.13).

(b) The Random Input:

The random input u(lT) to the system, with density function $p_u(u)$, is defined by eqn. (3.1.14) and its range satisfies eqn. (3.2.4). Eqn. (3.2.2) defines the input - output relationship of the filter. The statistics of the output may be evaluated as in the case when the input is deterministic.

(i) The conditional probability of z(Nl+k): z(Nl+k), given by eqn. (3.2.2a) is a function of both z(Nl+k-1) and u(lT). The probability $p_{z_o}(1)$ and $p_{z_o}(-1)$ are both equal to half. Therefore the probability of z(Nl+k), conditioned on u(lT) and z(Nl+k-1), may be evaluated.

$$z(Nl+K) = Sgn\ (u\ (lT) + b_1 z(Nl+K-1) + n(Nl+k))$$

Therefore,

$$p_{z_{Nl+k} \mid z_{Nl+k-1}, u(lT)}(1 \mid z(Nl+k-1), u(lT))$$

$$= \frac{1+a_o u(lT)+b_1 z(Nl+k-1)}{2} \qquad (3.2.31a)$$

$$p_{z_{Nl+k}\,|\,z_{Nl+k-1},u(lT)}(-1\,|\,z(Nl+k-1),u(lT)) = \frac{1-a_o u(lT)-b_1 z(Nl+k-1)}{2} \qquad (3.2.31b)$$

(ii) The Expected value of $y(lT)$:

$$E(y(lT)) = \frac{1}{N}\sum_{k=1}^{N} E(z(Nl+k))$$

Expected value of $z(Nl+k)$ is given by

$$\begin{aligned} E(Z(Nl+k)) &= E[\mathrm{sgn}(a_o u(lT)+n(Nl+k)+b_1 z(Nl+k-1)] \\ &= E[a_o u(lT)+b_1\ z(Nl+k-1)] \\ &= a_o E(u(lT))+b_1\ E(z(Nl+k-1)) \end{aligned} \qquad (3.2.3$$

By analogy with eqn. (3.2.17), (3.2.32) reduces to

$$E(z(Nl+k)) = a_o\frac{1-b_1^k}{1-b_1}E(u(lT))+a_o b_1^k\frac{(1-b_1^N)}{1-b_1}\cdot\sum_{r=1}^{l}[b_1^{N(r-1)}E(u(\overline{l-r}T))] \qquad (3.2.33)$$

Therefore,

$$E(y(lT)) = \frac{1}{N}\sum_{k=1}^{N}[a_o\frac{1-b_1^k}{1-b_1}\cdot\eta_u+b_1^k\frac{1-b_1^N}{1-b_1}\cdot\eta_u\cdot\sum_{r=1}^{l} b_1^{N(r-1)}.a_o]$$

$$= \frac{a_o\eta_u}{1-b_1}-\frac{a_o\eta_u}{1-b_1}\cdot\frac{1}{N}\cdot b_1\frac{(1-b_1^N)}{1-b_1}+\frac{a_o\eta_u}{N}\cdot[\cdot\frac{1-b_1^N}{1-b_1}(\sum_{r=1}^{l} b_1^{N(r-1)})\cdot b_1\frac{1-b_1^N}{1-b_1}] \qquad (3.2.34)$$

In the limit, as $N\to\infty$, $E(\hat{y}(lT))$ tends to

$$\lim_{N\to\infty} E(\hat{y}(lT)) = \frac{a_o\eta_u}{1-b_1} \qquad (3.2.35)$$

Therefore, when N tends to infinity, $E(\hat{y}(lT))$ agrees with $E(y(lT))$, (eqn. 3.1.14b) if $\eta_u=0$.

(iii) The Ensemble Autocorrelation function of $\hat{y}(lT)$:

The autocorrelation function of $\hat{y}(lT)$ is given by

$$R_{\hat{y}}(k,k+m)=E[\hat{y}(kT)\hat{y}(\overline{k+m}T)] \qquad (3.2.36)$$

$$= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} E(z(Nk+i)z(N\overline{k+m}+j))$$

Using eqn. (3.2.2a)

$$R_{\hat{y}}(k,k+m)= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} E[\mathrm{sgn}(a_o u(kT)+n(Nk+i)+b_1 Z(Nk+i-1)).$$

$$\mathrm{sgn}(a_o\ u(\overline{k+m}T)+n(N\overline{k+m}+j)+b_1\ (N\overline{k+m}+j-1))] \qquad (3.2.37)$$

For m=o:-

$$R_{\hat{y}}(k,k)= \frac{1}{N^2}[\sum_{i=1}^{N} E(z(Nk+i))^2 + \sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} E[\mathrm{sgn}(a_o u(kT)+n(Nk+i)$$

$$+b_1 Z(Nk+i-1)).\ \mathrm{sgn}(a_o u(kT)+n(Nk+j)+b_1 Z(Nk+j-1))]$$

$$= \frac{1}{N^2}[N+\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} E[(a_o u(kT)+b_1 z(Nk+i-1))(a_o u(kT)$$

$$+b_1 z(Nk+j-1))] \qquad (3.2.38)$$

$$= \frac{1}{N^2}\left[N+(N^2-N)a_o^2\ \sigma_u^2+(N-1)a_o b_1 \sum_{j=1}^{N} E(u(kT)z(Nk+j-1))+(N-1)\ .\right.$$

$$\left. a_o b_1 \sum_{i=1}^{N} E(u(kT)z(Nk+i-1))+b_1^2 \sum_{\substack{i=1\\ i\neq j}}^{N} \sum_{j=1}^{N} E(z(Nk+i-1)z(Nk+j-1))\right] \tag{3.2.39}$$

It is required to evaluate the cross-correlation function

$$R_{u,z}\ (kT,\overline{Nk+j-1}T)=E[u(kT)z(Nk+j-1)] \quad j=1,2\ldots,N \tag{3.2.40}$$

For $j=1$

$$E[u(kT)z(Nk+j-1)]=E[u(kT)z(N\overline{k-1}+N)\ ] \tag{3.2.41}$$

From the structure of the filter, $u(kT)$ and $z(N\overline{k-1}+N)$ are independent, the latter being a function of $u(\overline{k-1}T)$, $u(\overline{k-2}T),..u(oT)$, of which $u(kT)$ is independent. By definition $E(u(kT))=0$ hence (3.2.41) reduces to

$$E[u(kT)Z(N\overline{k-1}+N)] = E(u(kT)).E[Z(N\overline{k-1}+N))]=0 \tag{3.2.42}$$

For $2 \leq j \leq N$

$$E[u(kT)z(Nk+j-1)]=E[u(kT)\mathrm{sgn}\{a_o u(kT)+n(Nk+j-1)+b_1 z(Nk+j-2)\}]$$

$$=E[u(kT)(a_o u(kT)+b_1 z(Nk+j-2)]$$

$$=a_o\ \sigma_u^2+b_1\ E[u(kT).z(Nk+j-2)]$$

$$=a_o\ \sigma_u^2\ [1+b_1+b_1^2+\ldots+b_1^{j-2}\ ]$$

$$=a_o\ \sigma_u^2 . \frac{1-b_1^{j-1}}{1-b_1} \quad 2 \leq j \leq N \tag{3.2.43}$$

Substituting eqn. (3.2.43) in eqn. (3.2.39) we obtain

$$R_{\hat{y}}(k,k) = \frac{1}{N^2}\Big[N+(N^2-N).a_o^2\,\sigma_u^2+2(N-1)a_o b_1 \sum_{j=2}^{N} a_o\sigma_u^2 . \frac{1-b^{j-1}}{1-b_1}$$

$$+b_1^2 \sum_{\substack{i=1 \\ i\neq j}}^{N} \sum_{j=1}^{N} E[z(Nk+i-1)z(Nk+j-1)]\Big] \qquad (3.2.44)$$

$$= \frac{1}{N^2}\Big[N+(N^2-N)a_o^2\sigma_u^2+2(N-1)a_o^2 b_1\sigma_u^2\,\frac{1}{1-b_1} - \frac{b_1-b_1^N}{(1-b_1)^2}\Big]$$

$$+ b_1^2 \sum_{\substack{i=1 \\ i\neq j}}^{N} \sum_{j=1}^{N} E[z(Nk+i-1)z(Nk+j-1)]] \qquad (3.2.45)$$

In the limiting case, when $N\rightarrow\infty$, $R_{\hat{y}}(k,k)$ tends to

$$\lim_{N\rightarrow\infty} R_{\hat{y}}(k,k)=a_o^2\sigma_u^2+ \lim_{N\rightarrow\infty} \frac{b_1^2}{N^2} \sum_{\substack{i=1 \\ i\neq j}}^{N} \sum_{j=1}^{N} E[z(Nk+i-1)z(Nk+j-1)] +$$

$$2a_o^2\sigma_u^2 b_1(1-b_1)^{-1} \qquad (3.2.46)$$

Comparing eqn. (3.2.46) with eqn. (3.1.17) for m=o, we see that (3.2.46) contains other terms, apart from those contained in eqn. (3.1.17).

For $m\neq o$:-

$$R_{\hat{y}}(k,k+m)= \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} E[\mathrm{sgn}(a_o u(kT)+n(Nk+i)+b_1 z(Nk+i-1)).$$

$$\mathrm{sgn}(a_o u(\overline{k+m}T)+n(\overline{Nk+m}+j)+b_1 z(\overline{Nk+m}+j-1))]$$

$$(3.2.47)$$

Therefore $n(Nk+i)$ and $n(N\overline{k+m}+j)$ are independent $(m\neq o)$, hence

$$R_{\hat{y}}(k,k+m)=\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}E[a_o^2u(kT)u(\overline{k+m}T)+a_ob_1u(\overline{k+m}T)z(Nk+i-1)$$

$$+a_ob_1u(kT)z(N\overline{k+m}+j-1)+b_1^2\,z(Nk+i-1)z(N\overline{k+m}+j-1)] \qquad (3.2.48)$$

$$=\frac{1}{N^2}[a_ob_1.N.\sum_{i=1}^{N}E(u(\overline{k+m}T)z(Nk+i-1))+Na_ob_1\sum_{j=1}^{N}E(u(kT).$$

$$z(N\overline{k+m}+j-1)+b_1^2\sum_{i=1}^{N}\sum_{j=1}^{N}E(z(Nk+i-1)z(N\overline{k+m}+j-1))]$$

$$(3.2.49)$$

To evaluate $R_{\hat{y}}(k,k+m)$, therefore, the cross-correlation functions $E(u(\overline{k+m}T)z(Nk+i-1)$, $E(u(kT)z(N\overline{k+m}+j-1), i=1,\ldots,N,$ $j=1,\ldots,N$ need to be evaluated.

<u>$E[u(\overline{k+m}T)z(Nk+i-1)]$</u> , $i=1,2,\ldots,N;\ m\neq o$

$$E[u(\overline{k+m}T)\ z(Nk+i-1)]=0 \quad m\geq 1, \text{ for all } i$$

$$=a_ob_1^{i-1}.\sigma_u^2b_1^{N(|m|-1)}\cdot\frac{1-b_1^N}{1-b_1}$$

$$m\leq -1, \text{ for all } i. \qquad (3.2.50)$$

<u>$E[u(kT)z(N\overline{k+m}+j-1)]$</u> , $j=1,2,\ldots,N \quad m\neq o$

$$E[u(kT)z(N\overline{k+m}+j-1)]=o \quad m\leq -1, \text{ all } j$$

$$=a_o\sigma_u^2b_1^{j-1}.b_1^{N(m-1)}.\frac{1-b_1^N}{1-b_1}\ ;\ m\geq 1, \text{ all } j$$

$$(3.2.51)$$

Substituting (3.2.50) and (3.2.51) in (3.2.49) we obtain

$$R_{\hat{y}}(k,k+m)=\frac{1}{N^2}\,[a_o b_1 N.\sum_{i=1}^{N} a_o b_1^{i-1}.\sigma_u^2\, b_1^{N(|m|-1)}\,.\,\frac{1-b_1^N}{1-b_1}$$

$$+\, b_1^2 \sum_{i=1}^{N}\sum_{j=1}^{N} E(z(Nk+i-1)z(N\overline{k+m}+j-1))]$$

$$=\frac{1}{N^2}\,[a_o^2 b_1 N\sigma_u^2 b_1^{N(|m|-1)}\left(\frac{1-b_1^N}{1-b_1}\right)^2 + b_1^2 \sum_{i=1}^{N}\sum_{j=1}^{N}$$

$$E(z(Nk+i-1)z(N\overline{k+m}+j-1))] \qquad (3.2.52)$$

Eqn. (3.2.52) therefore is the recursive equation for evaluating $R_{\hat{y}}(k\ ,k+m)\big|_{m\neq o}$. In the limit, as $N\rightarrow\infty$ $R_{\hat{y}}(k,k+m)\big|_{m\neq o}\rightarrow o$, it is seen from eqns. (3.2.46) and (3.2.52) that the autocorrelation function of the output of the coarse-sampled-data filter of the first order does not equal that of the output of the linear digital filter, even with infinite averaging time ($N\rightarrow\infty$).

3.2.2 The Second Order Coarse-sampled-data Filter:

A linear second order digital filter is given by the difference equation:

$$y(lT) = a_o u(lT)+b_1 y(\overline{l-1}T)+b_2\, y(\overline{l-2}T)$$

For convenience, let b_1=o. Hence y(kT) is given by

$$y(lT) = a_o u(lT)+b_2 y(\overline{l-2}T) \qquad (3.2.53)$$

Kirlin's coarse sampled data filter replacing a linear second order digital filter is given by the equations.

$$Z(Nl+k)=\text{sgn}(a_o u(lT)+n(Nl+k)+b_2 z(Nl+k-2) \qquad (3.2.54a)$$

$$\hat{y}(lT) = \frac{1}{N} \sum_{k=1}^{N} z(Nl+k) \qquad (3.2.54b)$$

The input u(lT) is as defined in eqn. (3.1.14). The period of the dither n(Nl+k) is T' seconds, n(Nl+k) is uniformly distributed in the range (-1,+1) so that

$$\begin{aligned} a_o u(lT)_{max} + |b_2| &\leq 1 \\ a_o u(lT)_{min} - |b_2| &\geq -1 \end{aligned} \qquad (3.2.55)$$

$$p_n(n) = \frac{1}{2} \quad -1 \leq n \leq 1 \qquad (3.2.56)$$

Fig. 3.2.2 shows a schematic diagram of the second order coarse-sampled-data filter.

(a) <u>The Deterministic Input:</u> The deterministic input u(lT) is such that the eqn. (3.2.55) holds. The output of the coarse sampled data filter is given by eqn.s (3.2.54) .

(i) The probability of z(Nl+k): The initial values of z(Nl+k-1) and z(Nl+k-2) at l=o and k=1 are set such that the initial stored values z(o), z(-1) are each ± 1 with equal probability and are mutually independent i.e.

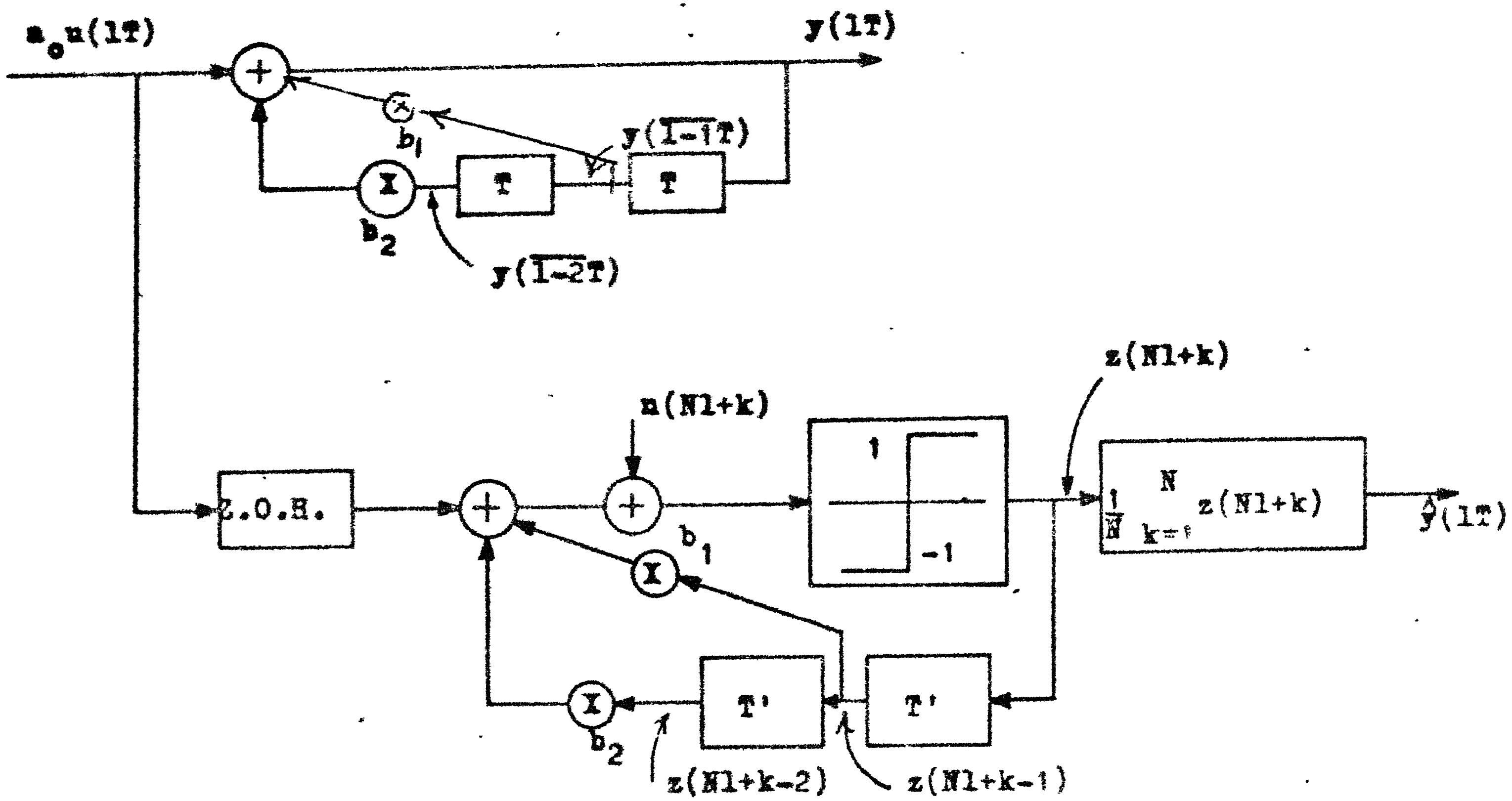


Fig. 3.2.2 Schematic of the second order linear digital filter and the second order coarse-sampled-data filter.

$$E(z(Nl+k))=E[\mathrm{sgn}(a_o u(lT)+n(Nl+k)+b_2 z(Nl+k-2))]$$

$$=a_o u(lT)+b_2 E[z(Nl+k-2)] \qquad (3.2.62)$$

$$=a_o u(lT).\ \frac{1-b_2^{\frac{k+1}{2}}}{1-b_2}+b_2^{\frac{k-1}{2}}\ .\ a_o.\ \frac{1-b_2^{\frac{N}{2}+1}}{1-b_2}\ .$$

$$\sum_{r=1}^{l} u(\overline{l-r}T)(b_2^{N/2})^{r-1} \quad \text{for odd k: even N} \quad (a)$$

$$=a_o u(lT).\ \frac{1-b_2^{\frac{k+1}{2}}}{1-b_2}+b_2^{\frac{k-1}{2}}\ .a_o\ \frac{1-b_2^{\frac{N+1}{2}}}{1-b_2}\ .$$

$$\sum_{r=1}^{l} u(\overline{l-r}T)(b_2^{\frac{N-1}{2}})^{r-1} \quad \text{for odd k; odd N} \quad (b)$$

$$=a_o u(lT).\ \frac{1-b_2^{\frac{k}{2}+1}}{1-b_2}+b_2^{k/2}.a_o.\ \frac{1-b_2^{\frac{N}{2}+1}}{1-b_2}\ .$$

$$\sum_{r=1}^{l} u(\overline{l-r}T).(b_2^{N/2})^{r-1} \quad \text{for even k; even N} \quad (c)$$

$$=a_o u(lT).\ \frac{1-b_2^{\frac{k}{2}+1}}{1-b_2}+b_2^{k/2}\ .a_o.\ \frac{1-b_2^{\frac{N+1}{2}}}{1-b_2}$$

$$\sum_{r=1}^{l} u(\overline{l-r}T)(b_2^{\frac{N-1}{2}})^{r-1} \quad \text{for even k; odd N} \quad (d)$$

$$(3.2.63)$$

For the two cases, (i) N is odd (ii) N is even $E(\hat{y}(lT))$ is given by

$$E(\hat{y}(lT)) = \frac{1}{N}\left[\sum_{k=1,3,5\ldots N} E(z(Nl+k)) + \sum_{k=2,4,\ldots,N-1} E(z(Nl+k))\right] N \text{ odd} \quad (a)$$

$$= \frac{1}{N}\left[\sum_{k=1,3,\ldots N-1} E(z(Nl+k)) + \sum_{k=2,4,\ldots,N} E(z(Nl+k))\right] N \text{ even} \quad (b)$$

(3.2.64)

Therefore,

$$E(\hat{y}(lT)) = \frac{1}{N}\left[\frac{a_o u(lT)}{1-b_2} . N - \frac{a_o b_2 u(lT)}{(1-b_2)^2} . (1+b_2-2b_2^{\frac{N+1}{2}}) + a_o . \frac{1-b_2^{\frac{N+1}{2}}}{(1-b_2)^2} (1+b_2-2b_2^{\frac{N+1}{2}}) \sum_{r=1}^{l} u(l-rT)(b_2^{\frac{N-1}{2}})^{r-1}\right] \text{odd } N \quad (a)$$

$$= \frac{1}{N}\left[\frac{a_o u(lT)}{1-b_2} . N - \frac{a_o b_2 u(lT)}{(1-b_2)^2} (1+b_2)(1-b_2^{N/2}) + a_o \frac{1-b_2^{\frac{N}{2}+1}}{(1-b_2)^2} . (1+b_2)(1-b_2^{N/2}) \sum_{r=1}^{l} u(\overline{l-r}T)(b_2^{\frac{N}{2}(r-1)}\right] \text{ even } N \quad (b)$$

(3.2.65)

In the limit as $N \rightarrow \infty$ for infinite averaging $E(\hat{y}(lT))$ tends to

$$\lim_{N\rightarrow\infty} E(\hat{y}(lT)) = \frac{a_o u(lT)}{1-b_2} \quad \text{for N either odd or even} \quad (3.2.66)$$

Comparing the RHS of eqn. (3.2.66) with the o/p of the corresponding second order linear digital filter (3.1.50), it is seen that $\lim_{N\rightarrow\infty} E(\hat{y}(lT))$ is not close to the desired value $\hat{\hat{y}}(lT)$.

(ii) The Time Autocorrelation of $\hat{y}(lT)$: for the second order coarse sampled data filter is defined in a manner similar to that for the first order filter.

$$R_{\hat{y}}(m) = \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} \hat{y}(kT)\hat{y}(\overline{k+m}T) \tag{3.2.67}$$

$$= \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} z(Nk+i)Z(z\overline{k+m}+j) \tag{3.2.68}$$

$$E(R_{\hat{y}}(m)) = \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} E(z(Nk+i)z(N\overline{k+m}+j)) \tag{3.2.69}$$

$$= \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} E[sgn(a_o u(kT)+n(Nk+i)$$

$$+ b_2 z(Nk+i-2)).sgn(a_o u(\overline{k+m}T)+n(N\overline{k+m}+j)+b_2 z(N\overline{k+m}+j-2))] \tag{3.2.70}$$

$E(R_{\hat{y}}(m))$ can be expressed in the form of recursive equations.

<u>For m=o</u>

$$E(R_{\hat{y}}(o)) = \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} \frac{1}{N^2} [N+(N^2-N)a_o^2u^2(kT)+a_ob_2u(kT)(N-1)$$

$$\left\{\sum_{i=1}^{N} E(Z(Nk+i-2))+\sum_{j=1}^{N} E(Z(Nk+j-2))\right\} +b_2^2 \sum_{i=1}^{N} \sum_{j=1}^{N}$$

$$E(Z(Nk+j-2)Z(Nk+i-2))] \tag{3.2.71}$$

Depending on the nature of N, the appropriate two equations of equation (3.2.63) may be substituted in eqn. (3.2.71), to obtain $E(R_y^{\wedge}(m))$ by expanding the last term of eqn. (3.2.71) recursively.

For m≠o : From eqn. (3.2.70) for m≠o,

$$E(R_y^{\wedge}(m)) = \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} E[(a_o u(kT)+b_2 z(nK+i-2))$$

$$(a_o u(\overline{k+m}T)+b_2 z(N\overline{k+m}+j-2))]$$

$$= \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} \frac{1}{N^2} [N^2 a_o^2 u(kT)u(\overline{k+m}T)+N\ a_o b_2\ u(\overline{k+m}T)$$

$$\sum_{i=1}^{N} E(z(Nk+i-2))+u(kT) \sum_{j=1}^{N} E(z(N\overline{k+m}+j-2))+$$

$$b_2^2 \sum_{i=1}^{N} \sum_{j=1}^{N} E(z(Nk+i-2)z(N\overline{k+m}+j-2))] \qquad (3.2.72)$$

As N is odd or even, the appropriate substitutions may be made from (3.2.63) in eqn. (3.2.72) and the last term in eqn. (3.2.72) expanded recursively to give $E(R_y^{\wedge}(m))$.

(b) The Random Input: to the second order coarse-sampled-data filter u(lT) is white, stationary and independent as described in eqn. (3.2.14). In addition, its range is such that eqn. (3.2.55) is satisfied. The statistics of the output $\hat{y}(lT)$ of this filter are derived thus:

(i) The conditional probability of z(Nl+K): The values of z(Nl+k-1), z(Nl+k-2) are initialized at l=o, k=1 such that z(o), z(-1) are ± 1 with equal probability i.e. eqn. (3.2.57) holds. Eqn. (3.2.54a) gives z(Nl+k), a function of u(lT), n(Nl+k) and z(Nl+k-2). Hence the probability of z(Nl+k) conditioned on both u(lT) and z(Nl+k-2) is given by

$$p_{z_{Nl+k} \mid z_{Nl+k-2},u(lT)}(1 \mid z(Nl+k-2),u(lT)) = \frac{1+a_o u(lT)+b_2 z(Nl+k-2)}{2}$$

(3.2.73a)

$$p_{z_{Nl+k} \mid z_{Nl+k-2},u(lT)}(-1 \mid z(Nl+k-2),u(lT)) = \frac{1-a_o u(lT)-b_2 z(Nl+k-2)}{2}$$

(3.2.73b)

The unconditioned probability of z(Nl+k) may be obtained from its expected value, evaluated next.

(ii) The Expectation of $\hat{y}(lT)$: Using eqn. (3.2.54)

$$E(\hat{y}(lT)) = \frac{1}{N} \sum_{k=1}^{N} E(z(Nl+k)) \qquad (3.2.74)$$

$$E(z(Nl+k)) = E[\mathrm{sgn}(a_o u(lT)+n(Nl+k)+b_2 Z(Nl+k-2))]$$

$$= a_o \eta_u + b_2 E[z(Nl+k-2)] \qquad (3.2.75)$$

The result is as given in eqn. (3.2.63), u(lT) and $u(\overline{l-r}T)$ being replaced by η_u, the substitution made as before:-

$$E(\hat{y}(lT)) = \frac{\eta_u}{N}\left[\frac{a_oN}{1-b_2} - \frac{a_ob_2}{(1-b_2)^2}\cdot(1+b_2-2b_2^{\frac{N+1}{2}}) + a_o\cdot\frac{1-b_2^{\frac{N+1}{2}}}{(1-b_2)^2}\right.$$

$$\left.(1+b_2-2b_2^{\frac{N+1}{2}})\sum_{r=1}^{l} b_2^{\frac{(N-1)(r-1)}{2}}\right] \quad \text{odd } N$$

$$= \frac{\eta_u}{N}\left[\frac{a_oN}{1-b_2} - \frac{a_ob_2}{(1-b_2)^2}\cdot(1+b_2)(1-b_2^{N/2}) + a_o\frac{1-b_2^{\frac{N}{2}+1}}{(1-b_2)^2}\right.$$

$$\left.(1+b_2)(1-b_2^{N/2})\sum_{r=1}^{l} b_2^{\frac{N}{2}(r-1)}\right] \quad \text{even } N$$

$$(3.2.76)$$

$$\lim_{N\to\infty} E(\hat{y}(lT)) = \frac{a_o\eta_u}{1-b_2} \qquad (3.2.77)$$

If $u(lT)$ is zero mean, RHS of (3.2.77) is zero, tallying with $E(\hat{y}(kT))$ obtained from equation (3.1.50). But if $\eta_u \neq o$ (3.2.77) is seen to be greater.

Evaluation of probability of $z(Nl+k)$:

Let $p_{z_{Nl+K}}(1) = p$

$$E(z) = p + (-1)(1-p)$$

$$p_{z_{Nl+K}}(1) = \frac{(E(z)+1)}{2} \qquad (3.2.78)$$

Since $E(z)$ is already available from eqn. (3.2.75) and (3.2.63), 'p' may be evaluated.

(iii) The Ensemble Autocorrelation of y(1T) is given by

$$R_{\hat{y}}(k,k+m)=E(\hat{y}(kT)\hat{y}(\overline{k+m}T))$$

$$=\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}E(z(Nk+i)z(Nk+m+j)) \qquad (3.2.79)$$

m=o:-

$$R_{\hat{y}}(k,k)=\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}E(z(Nk+i)z(Nk+j)) \qquad (3.2.80)$$

$$=\frac{1}{N^2}[N+\sum_{\substack{i=1\\i\neq j}}^{N}\sum_{j=1}^{N}E(z(Nk+i)z(Nk+j))] \qquad (3.2.81)$$

$$=\frac{1}{N^2}[N+\sum_{\substack{i=1\\i\neq j}}^{N}\sum_{j=1}^{N}E[sgn(a_ou(kT)+n(Nk+i)+b_2z$$

$$(Nk+i-2)).Sgn(a_ou(kT)+n(Nk+j)+b_2z(Nk+j-2))]$$

$$=\frac{1}{N^2}[N+(N^2-N)a_o^2\sigma_o^2+2a_ob_2(N-1)\sum_{i=1}^{N}E(u(kT)$$

$$z(Nk+i-2))+b_2^2\sum_{\substack{i=1\\i\neq j}}^{N}\sum_{j=1}^{N}E(z(Nk+i-2)z(Nk+j-2))]$$

$$(3.2.82)$$

For odd N or even N, eqn. (3.2.81) reduces to

$$R_{\hat{y}}(k,k)=\frac{1}{N^2}[N+(N^2-N)a_o^2\sigma_u^2+2a_ob_2(N-1)\frac{a_o\sigma_2^2}{1-b_2}(N-2)-\frac{a_c\sigma_2^2}{(1-b_2)^2}.$$

$$(b_2+b_2^2-2b_2^{\frac{N+1}{2}})+b_2^2\sum_{\substack{i=1\\i\neq j}}^{N}\sum_{j=1}^{N}E(z(NK+i-2)z(Nk+j-2))]$$

for odd N, (3.2.83a)

$$= \frac{1}{N^2}[N+(N^2-N)a_o^2\sigma_u^2+2a_o b_1(N-1)\left\{\frac{a_o\sigma_u^2}{1-b_2}(N-2)-\frac{a_o\sigma_u^2}{(1-b_2)^2}\cdot(1+b_2)\right.$$

$$\left.(b_2-b_2^{N/2})^2\right\}+b_2^2\sum_{\substack{i=1\\ i\neq j}}^{N}\sum_{j=1}^{N}E(z(Nk+i-2)z(Nk+j-2))] \text{ for even } N$$

(3.2.83b)

Thus it can be seen that with $N\rightarrow\infty$ for infinite averaging, eqn. (3.2.83) reduces to the recursive equation:

$$\lim_{N\rightarrow\infty} R_{\hat{y}}(k,k)= a_o^2\sigma_u^2+\frac{2a_o^2 b_1\sigma_u^2}{1-b_2}+\lim_{N\rightarrow\infty} b_2^2\sum_{\substack{i=1\\ i\neq j}}^{N}\sum_{j=1}^{N}$$

$$E(z(Nk+i-2)\ z(Nk+j-2)) \qquad (3.2.84)$$

For $m\neq o$ eqn. (3.2.79) can be rewritten as,

$$R_{\hat{y}}(k,k+m)=\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}E[(\text{sgn}(a_o u(kT)+n(Nk+i)+b_2\ z(Nk+i-2)).$$

$$(\text{sgn}(a_o u(\overline{k+m}T)+n(N\overline{k+m}+j)+b_2\ z(Nk+j-2))] \qquad (3.2.85)$$

$$=\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}E[(a_o u(kT)+b_2 z(Nk+i-2)(a_o u(\overline{k+m}T)+b_2 z(N\overline{k+m}+j-2))$$

(3.2.86)

$$=\frac{1}{N^2}[a_o b_2 N\sum_{i=1}^{N}E(u(k+mT)z(Nk+i-2))+a_o b_2 N\sum_{j=1}^{N}E(u(kT)$$

$$z(Nk+m+j-2)+b_2^2\sum_{i=1}^{N}\sum_{j=1}^{N}E(z(Nk+i-2)z(N\overline{k+m}+j-2))]$$

(3.2.87)

This recursive equation may be evaluated for N odd or even and for different values of m.

Summary: The coarse-sampled-data filter using an RC averager, has been introduced by Kirlin [7] to replace a linear digital filter implementing by a particular difference equation. The expected value of its output $\hat{y}(lT)$ is said to be equal to that of the output of the linear digital filter. A saving in hardware requirements is made with the use of this non-linear filter. In this section (3.2) the first and second order coarse sampled data filter, using digital counter averagers, are discussed, and the statistics of their outputs are compared with those of the outputs of the corresponding linear digital filters. It is seen that with this form of the coarse sampled data filter, the statistics of $\hat{y}(lT)$ do not approximate those of $y(lT)$.

3.3 The N-bit Feedback Coarse-Sampled-Data Filter:

In the previous section, the coarse-sampled-data filter was discussed and it was shown that it cannot be used to replace the linear digital filter, as such. A modification of Kirlin's filter leads to the N-bit feedback coarse-sampled-data filter, the mean and covariance of the output of which can be made to match, as closely as desired, the mean and covariance of the output of the linear digital

filter. The filter configuration is developed as follows:

The output $\hat{y}(lT)$ of the coarse-sampled-data filter is obtained by first resetting the digital counter averager to zero. The counter then upcounts or downcounts as $z(Nl+k)$ is (+ve) or (-ve) respectively, for $k=1,2,...,N$. Subsequently the sum stored in the counter is normalized by dividing by N. If it can be ensured that over $k=1,...,N$ the outputs of the comparator, $z(Nl+k)$'s, are identically distributed and that $E[z(Nl+k)]=E(y(lT))$, then the expected value of $\hat{y}(lT)$ equals the expected value of $y(lT)$. This objective being satisfied, the respective autocorrelation functions of the outputs of the two filters, the linear and the non-linear, may be compared.

To ensure identical distributions of $z(Nl+k)$, $k=1,...,N$, (the outputs of the comparator), the feedback for a filter of order M, contains, not M elements as in the case of Kirlin's coarse-sampled-data filter, but MN elements where N relates the dither rate T' and the input/output sampling rate T' ($T'=T/N$). The MN elements of the feedback, $z(o)$, $z(-1)$, ..., $z(-MN+1)$ are set at $l=o$ to +1 or -1 with equal probability and are mutually independent. Hence for a linear digital filter of order M given by

$$y(lT) = \sum_{i=o}^{M} a_i u(\overline{l-i}T) + \sum_{j=1}^{M} b_j y(\overline{l-j}T) \quad , \qquad (3.3.1)$$

the N-bit feedback coarse-sampled-data filter is given by the relations

$$z(Nl+k) = \text{sgn}\left(\sum_{i=o}^{M} a_i u(\overline{l-i}T) + n(Nl+k) + \sum_{j=1}^{M} z(N\overline{l-j}+k) \right) \qquad (3.3.2a)$$

$$\hat{y}(lT) = \frac{1}{N} \sum_{k=1}^{N} z(Nl+k) \qquad (3.3.2b)$$

As in sec. 3.2, the dither, lying in the range (-1,+1), is such that

$$\left[\sum_{i=o}^{M} a_i u(\overline{l-i}T) \right]_{max} + \sum_{i=1}^{M} \left| b_i \right| \leqslant +1 \qquad (3.3.3a)$$

and

$$\left[\sum_{i=o}^{M} a_i u(\overline{l-i}T) \right]_{min} - \sum_{i=1}^{M} \left| b_i \right| \geqslant -1 \qquad (3.3.3b)$$

Choosing such a range for 'n' ensures that the input to the comparator, over a period T, is (+ve) as well as (-ve), avoiding saturation of the output of the averager ,which makes an averaging of the output of the comparator meaningless.

The dither noise n(Nl+k) is independent between samples, is uncorrelated and is uniformly distributed in the range (-1,1), i.e. if $p_n(n)$ is the probability density of the dither,

$$p_n(n) = \frac{1}{2} \quad -1 \leq n \leq 1 \tag{3.3.4}$$

The output statistics are evaluated for the first and second order filters, for the cases of deterministic and random inputs.

3.3.1 The First Order N-bit Feedback Filter:

The first order linear digital filter follows the difference equation

$$y(lT) = a_0 u(lT) + b_1 y(\overline{l-1}T) \tag{3.3.5}$$

The corresponding N-bit feedback filter(Fig. (3.3.1) is given by the equations

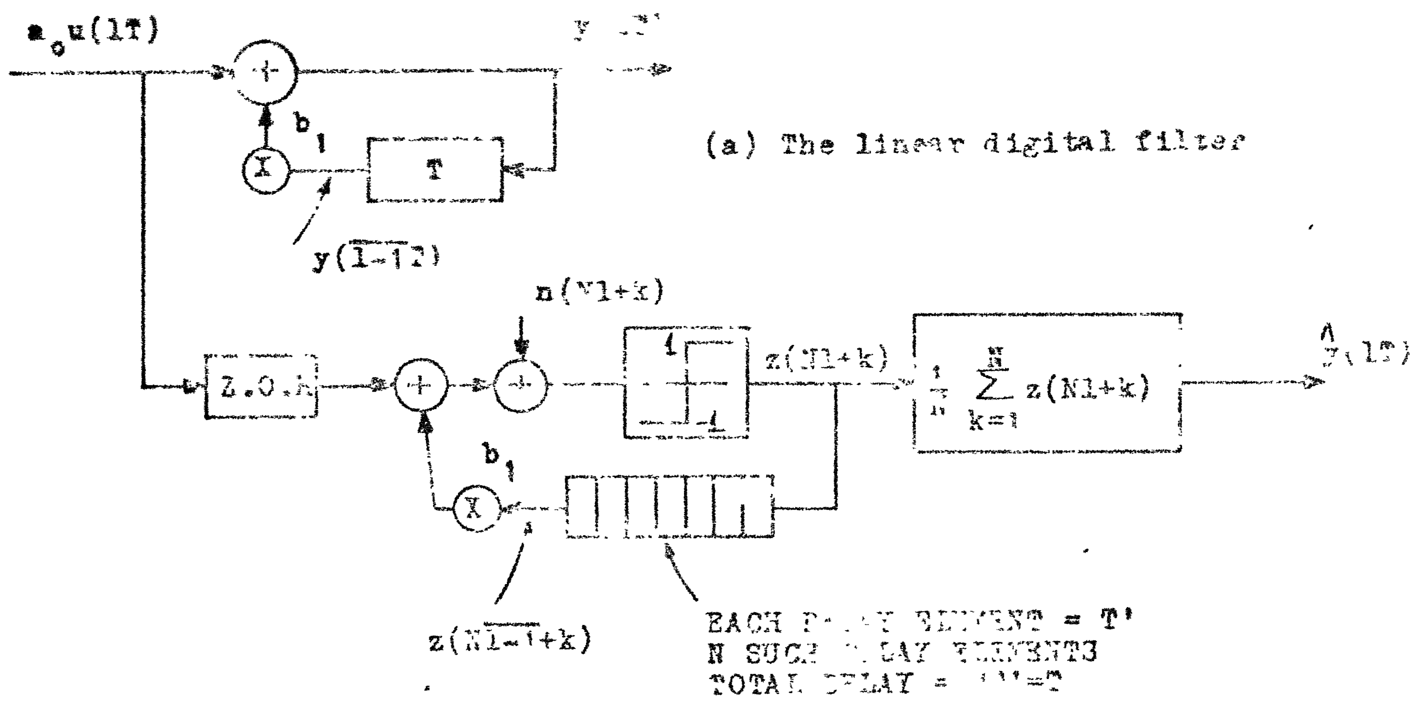
$$z(Nl+k) = \text{sgn}(a_0 u(lT) + n(Nl+k) + b_1 z(N\overline{l-1}+k) \tag{3.3.6a}$$

$$\hat{y}(lT) = \frac{1}{N} \sum_{k=1}^{N} z(Nl+k) \tag{3.3.6b}$$

The added dither $n(Nl+k)$ follows equations (3.3.4), with $M=1$.

(a) The Deterministic Input: Let $u(lT)$ be deterministic, its range obeying eqn. (3.3.3).

(i) The Probability of $z(Nl+k)$: The probability of $z(Nl+k)$, conditioned on $z(N\overline{l-1}+k)$ is



(a) The linear digital filter

(b) The N-bit Feedback coarse-sampler-rate filter

Fig. 3.3.1

$$p_{z_{Nl+k} \mid z_{N\overline{l-1}+k}}\left(1 \mid z(N\overline{l-1}+k)\right)= \frac{1+a_o u(lT)+b_1 z(N\overline{l-1}+k)}{2} \tag{3.3.7}$$

$$p_{z_{Nl+k} \mid z_{N\overline{l-1}+k}}\left(-1 \mid z(N\overline{l-1}+k)\right)= \frac{1-a_o u(lT)-b_1 z(N\overline{l-1}+k)}{2} \tag{3.3.8}$$

The unconditional probability $p(z(Nl+k))$ is given by the recursive equation

$$p_{z_{Nl+k}}(1) = \frac{1+a_o u(lT)-b_1}{2} + b_1\, p_{z_{N\overline{l-1}+k}}(1)$$

$$= \sum_{i=o}^{l} \frac{(1+a_o u(iT)-b_1)}{2}\, b_1^{l-i} + \frac{b_1^{l+1}}{2} \tag{3.3.9a}$$

Similarly,

$$p_{z_{Nl+k}}(-1)= \frac{1-a_o u(lT)-b_1}{2} + b_1\, p_{z_{N\overline{l-1}+k}}(-1)$$

$$= \sum_{i=o}^{l} \frac{1-a_o u(iT)-b_1}{2} \cdot b_1^{l-i} + \frac{b_1^{l+1}}{2} \tag{3.3.9b}$$

Thus $z(Nl+k)$, $k=1,2,\ldots N$, are identically distributed, and independent of 'k'.

(ii) The Expected Value of $\hat{y}(lT)$:

Eqn. (3.3.6) gives the input-output relation for the first order N-bit feedback coarse-sampled-data filter. Using eqn. (3.3.6b)

$$E(\hat{y}(lT)) = \frac{1}{N} \sum_{k=1}^{N} E[z(Nl+k)] \qquad (3.3.10)$$

From eqn. (3.3.6ba)

$$E(z(Nl+k)) = E[\mathrm{Sgn}(a_o u(lT)+n(Nl+k)+b_1 z(N\overline{l-1}+k))] \qquad (3.3.11)$$

$$= E[a_o u(lT)+b_1 z(N\overline{l-1}+k)]$$

$$= a_o u(lT)+b_1 E(z(N\overline{l-1}+k)) \qquad (3.3.12)$$

$$\therefore \quad E(z(Nl+k)) = a_o \sum_{i=o}^{l} u(iT) b_1^{l-i} \qquad (3.3.13)$$

We see that $E(z(Nl+k))$ is independent of k. Substituting $E(z(Nl+k))$ in (3.3.10) we obtain

$$E(\hat{y}(lT)) = \frac{1}{N} \sum_{k=1}^{N} [a_o \sum_{i=o}^{l} u(iT) b_1^{l-i}]$$

$$= a_o \sum_{i=o}^{l} u(iT) b_1^{l-i} \qquad (3.3.14)$$

Comparing eqn. (3.3.24) with eqn. (3.1.9) we see that the expected value of $\hat{y}(lT)$ equals $y(lT)$, the output of the linear digital filter, for a deterministic input $u(lT)$. Comparing eqn. (3.3.13) with (3.1.8), we see, too, that

$$E(z(Nl+k)) = y(lT) \qquad (3.3.15)$$

(iii) The Time Autocorrelation of $\hat{y}(lT)$:

Let the time autocorrelation function of $\hat{y}(lT)$ be denoted by $R_{\hat{y}}(m)$

$$R_{\hat{y}}(m) \stackrel{\Delta}{=} \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} \hat{y}(kT)\hat{y}(\overline{k+m}T) \qquad (3.3.16)$$

$R_{\hat{y}}(m)$ is a function of the added dither $n(Nl+k)$. Hence

$$E(R_{\hat{y}}(m)) = \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} E(\hat{y}(kT)\hat{y}(k+mT)) \qquad (3.3.17)$$

To evaluate $E(\hat{y}(kT)\ \hat{y}(\overline{k+m}T))$

For m=o:-

$$E(\hat{y}(kT)\hat{y}(kT)) = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} E(z(Nk+i)z(Nk+j))$$

$$= \frac{1}{N^2} \left[\sum_{i=1}^{N} E(z^2(Nk+i)) + \sum_{\substack{i=1 \\ i\neq j}}^{N} \sum_{j=1}^{N} E(z(Nk+i)z(Nk+j)) \right] \qquad (3.3.18)$$

$$\begin{aligned} E(z(Nk+i)z(Nk+j))\Big|_{i\neq j} &= E[sgn(a_o u(kT)+n(Nk+i)+b_1 z(N\overline{k-1}+i)). \\ &\quad Sgn(a_o u(kT)+n(Nk+j)+b_1 z(N\overline{k-1}+j))] \\ &= a_o^2\ u^2(kT)+a_o u(kT).b_1[E(z(N\overline{k-1}+i)+ \\ &\quad z(N\overline{k-1}+j))]+b_1^2 E(z(N\overline{k-1}+j)z(N\overline{k-1}+i)) \\ &= a_o^2 u^2(kT)+2a_o u(kT).b_1\ y(\overline{k-1}T)+b_1^2. \\ &\quad E[z(N\overline{k-1}+j).z(N\overline{k-1}+i)] \\ &= y^2(kT) \end{aligned} \qquad (3.3.19)$$

$$E(\hat{y}(kT)\hat{y}(kT)) = \frac{1}{N^2} \left[N+(N^2-N)y^2(kT)\right] \qquad (3.3.19b)$$

In the limit as $N\to\infty$,

$$\lim_{N\to\infty} E(\hat{y}(kT))^2 = y^2(kT) \qquad (3.3.20)$$

For $m\neq o$.

$$E(\hat{y}(kT)\hat{y}(\overline{k+m}T)) = \frac{1}{N^2} \sum_{i=1}^{N}\sum_{j=1}^{N} E(z(Nk+i)z(N\overline{k+m}+j)) \qquad (3.3.21a)$$

$$= \frac{1}{N^2}\left[\sum_{i=1}^{N} E(z(Nk+i)z(N\overline{k+m}+i)) + \sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} E(z(Nk+i)z(N\overline{k+m}+j)\right]$$

$$E(\hat{y}(kT)\hat{y}(\overline{k+m}T))\Big|_{m>o} = \frac{1}{N^2}\left[N(b_1^m+a_o y(kT)\sum_{i=o}^{m-1} b_1^i\, u(\overline{k+m-i})\right.$$

$$\left.+(N^2-N)y(kT)y(\overline{k+m}T)\right] \qquad (3.3.21b)$$

$$E(y(kT)y(\overline{k+m}T))\Big|_{m<o} = \frac{1}{N^2}\left[N(b_1^{|m|} + a_o y(\overline{k+m}T) \sum_{i=o}^{|m|-1} b_1^i u(\overline{k-i}T)\right.$$

$$\left.+(N^2-N)\, y(kT)y(\overline{k+m}T)\right] \qquad (3.3.21c)$$

Therefore, in the limit as $N\to\infty$, for infinite averaging and for all m $E(\hat{y}(kT)\hat{y}(\overline{k+m}T))$ tends to

$$\lim_{N\to\infty} E(\hat{y}(kT)\hat{y}(\overline{k+m}T)) = y(kT)y(\overline{k+m}T) \qquad (3.3.22)$$

In the limit, as $N \to \infty$, the expected value of $R_{\hat{y}}(m)$ (eqn. 3.3.17) tends to

$$\lim_{N \to \infty} E(R_{\hat{y}}(m)) = \lim_{K \to \infty} \frac{1}{K+1} \sum_{k=0}^{K} y(kT)y(\overline{k+m}T) \qquad (3.3.23)$$

But by definition (eqn. 3.1.11), the RHS of (3.3.23) is the time autocorrelation function of y(lT).

$$\lim_{N \to \infty} E(R_{\hat{y}}(m)) = R_y(m) \qquad (3.3.24)$$

This implies that if the filter be allowed to run for an infinite length of time to obtain each sample of the output $\hat{y}(lT)$, the expected value of the time autocorrelation function of its output $\hat{y}(lT)$ shall equal the time autocorrelation function of the output y(lT) of the linear digital filter.

(b) <u>The Random Input</u>: Let u(lT), the random input to the filter, be defined as in eqns. (3.1.15). The input-output relationship of the first order N-bit feedback coarse sampled data filter is given by the eqns. (3.3.2) and the range of u(lT) satisfies eqn. (3.3.3). The statistics of the output $\hat{y}(lT)$ may be evaluated thus:

(i) The Probability of z(Nl+k): z(Nl+k) is given by eqn. (3.3.6a). It is a function of n(Nl+k), u(lT) and $z(\overline{Nl-1}+k)$

The random input u(lT) is independent between samples, hence u(lT) and $z(\overline{Nl+r}+s)$ $r<0$, all s are independent. The probability of z(Nl+k), therefore, conditioned on u(lT) and $z(\overline{Nl-1}+k)$, is given by

$$p_{z_{Nl+k}\mid z_{\overline{Nl-1}+k},u_{lT}}(1\mid z(\overline{Nl-1}+k),u(lT))=\frac{1+a_o u(lT)+b_1 z(\overline{Nl-1}+k)}{2}$$

(3.3.25a)

$$p_{z_{Nl+k}\mid z_{\overline{Nl-1}+k},u_{lT}}(-1\mid z(\overline{Nl-1}+k),u(lT))=\frac{1-a_o u(lT)-b_1 z(\overline{Nl-1}+k)}{2}$$

(3.3.25b)

The probability of z(Nl+k) conditioned on $z(\overline{Nl-1}+k)$ alone, is therefore given by

$$p(z(Nl+k)\mid z(\overline{Nl-1}+k))=\int_u p(z(Nl+k)\mid z(\overline{Nl-1}+k),u(lT))\,.\,p(u(lT))\,.\,du(lT) \qquad (3.3.26)$$

$$p_{z_{Nl+k}\mid z_{\overline{Nl-1}+k}}(1\mid z(\overline{Nl-1}+k)) = \frac{1+a_o\eta_u+b_1 z(\overline{Nl-1}+k)}{2}$$

(3.3.27a)

$$p_{z_{Nl+k}\mid z_{\overline{Nl-1}+k}}(-1\mid z(\overline{Nl-1}+k))=\frac{1-a_o\eta_u-b_1 z(\overline{Nl-1}+k)}{2} \qquad (3.3.27b)$$

where $\eta_u = E(u)$ (3.3.27c)

In the form of recursive equations, p(z(Nl+k)) is given by

$$p_{z_{Nl+k}}(1) = \frac{1+a_o\eta_u-b_1}{2} + b_1\ p_{z_{N\overline{l-1}+k}}(1) \tag{3.3.28a}$$

$$= \frac{1+a_o\eta_u-b_1}{2} \sum_{i=o}^{l} b_1^{l-i} + \frac{b_1^{l+1}}{2} \tag{3.3.28b}$$

Similarly,

$$p_{z_{Nl+k}}(-1) = \frac{1-a_o\eta_u-b_1}{2} + b_1\ p_{z_{N\overline{l-1}+k}}(-1) \tag{3.3.29a}$$

$$= \frac{1-a_o\eta_u-b_1}{2} + \sum_{i=o}^{l} b_1^{l-i} + \frac{b_1^{l+1}}{2} \tag{3.3.29b}$$

Eqns. (3.3.28b) and (3.3.29b) show that p(z(Nl+k)) is independent of k, and that z(Nl+k), k=1,...,N are identically distributed.

(ii) The expected value of $\hat{y}(lT)$:

$\hat{y}(lT)$ is given by eqns. (3.3.6), u(lT) being random. The expected value of $\hat{y}(lT)$ may be evaluated as follows.

$$E(\hat{y}(lT)) = \frac{1}{N} \sum_{k=1}^{N} E[z(Nl+k)] \tag{3.3.30}$$

From eqn. (3.3.6a)

$$E(z(Nl+k)) = E[\mathrm{Sgn}(a_o u(lT)+n(Nl+k)+b_1 z(N\overline{l-1}+k)] \tag{3.3.31}$$

$$= E[a_o u(lT) + b_1\ z(N\overline{l-1}+k)]$$

$$= a_o\eta_u \sum_{i=o}^{l} b_1^{l-i} \tag{3.3.32}$$

[illegible]

[illegible]

$$E(\hat{y}(lT)) = \frac{1}{N}\sum_{k=1}^{N}\left(a_o\eta_u \sum_{i=o}^{l} b_1^{\,l-i}\right) \qquad [illegible]$$

$$= a_o\eta_u \sum_{i=o}^{l} b_1^{\,l-i} \qquad (3.3.\ [illegible])$$

Comparing eqns. (3.3.34) and (3.1.16), we see that the [illegible] of (3.3.34) is equal to $E(y(lT))$. Hence the expected values of $\hat{y}(lT)$ and $y(lT)$ are equal, even if $\eta_u \neq o$. We also observe that the RHS of eqn. (3.3.32) is equal to $E(y(lT))$. Therefore,

$$E(z(Nl+k)) = E(y(lT)) \qquad (3.3.35)$$

(iii) The Ensemble Autocorrelation of $\hat{y}(lT)$: is defined as

$$R_{\hat{y}}(k,k+m) \triangleq E[\hat{y}(kT)\hat{y}(\overline{k+m}T)] \qquad (3.3.36)$$

$$= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} E(z(Nk+i)z(N\overline{k+m}+j)) \qquad (3.3.37)$$

For m=o:-

$$R_{\hat{y}}(k,k) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} E(z(Nk+i)z(Nk+j)) \qquad (3.3.38)$$

$$= \frac{1}{N^2}\left[\sum_{i=1}^{N} E(z^2(Nk+i)) + \sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} E(z(Nk+i)z(Nk+j))\right] \qquad (3.3.39)$$

It is known that $z(Nk+i)$, $z(Nk+j)$ $i \neq j$ for all i,j are identically distributed, hence the second term in (3.3.39) may be evaluated thus:

$$\sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} E[z(Nk+i)z(Nk+j)] = (N^2-N)E[z(Nk+i)z(Nk+j)]_{i \neq j} \tag{3.3.40}$$

$$= (N^2-N)E[\mathrm{sgn}(a_o u(kT)+n(Nk+i)+b_1 z(N\overline{k-1}+i)).$$

$$\mathrm{Sgn}(a_o u(kT)+n(Nk+j)+b_1 z(N\overline{k-1}+j))]_{i \neq j} \tag{3.3.41}$$

$$= (N^2-N)E[(a_o u(kT)+b_1 z(N\overline{k-1}+i))(a_o u(kT)+b_1 z(N\overline{k-1}+j))]_{i \neq j} \tag{3.3.42}$$

$$=(N^2-N)[a_o^2\sigma_u^2+a_o b_1 . E\left\{u(kT)z(N\overline{k-1}+i)+u(kT)z(N\overline{k-1}+j)\right\}$$

$$+b_1^2 \; E(z(N\overline{k-1}+j)z(N\overline{k-1}+i))]_{i \neq j}$$

$$= (N^2-N)[a_o^2\sigma_u^2+b_1^2 \; E(z(N\overline{k-1}+i)z(N\overline{k-1}+j))]_{i \neq j} \tag{3.3.43}$$

$$= (N^2-N) \; a_o^2\sigma_u^2 \; \frac{(1-b_1^{2k+2})}{1-b_1^2} \tag{3.3.44}$$

$$\therefore \; R_{\hat{y}}(k,k) = \frac{1}{N^2} \left[N+(N^2-N).a_o^2\sigma_u^2 . \frac{1-b_1^{2k+2}}{1-b_1^2}\right] \tag{3.3.45}$$

In the limit, as $N \to \infty$ for infinite averaging,

$$\lim_{N \to \infty} R_{\hat{y}}(k,k) = a_o^2\sigma_u^2 \cdot \frac{1-b^{2k+2}}{1-b_1^2} \tag{3.3.46}$$

For $m \neq o$

$$R_{\hat{y}}(k,k+m) = \frac{1}{N^2}\left[\sum_{i=1}^{N} E(z(Nk+i)z(N\overline{k+m}+i)) + \sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} E(z(Nk+i)z(N\overline{k+m}+j))\right] \qquad (3.3.47)$$

$$= \frac{1}{N^2}\left[Nb_1^m + (N^2-N)b_1^m . a_o^2\sigma_u^2 . \frac{1-b_1^{2k+2}}{1-b_1^2}\right] \quad m > o \qquad (3.3.48a)$$

$$= \frac{1}{N^2}\left[N\, b_1^{|m|} + (N^2-N)b_1^{|m|}\, a_o^2\sigma_u^2 . \frac{1-b_1^{2(k+m)+2}}{1-b_1^2}\right] \quad m < o \qquad (3.3.48b)$$

In the limit, as $N \to \infty$ for infinite averaging, $R_{\hat{y}}(k,k+m)$ tends to a limiting value:

$$\lim_{N\to\infty} R_{\hat{y}}(k,k+m) = a_o^2\sigma_u^2(1-b_1^2)^{-1} \quad m = o \qquad (3.3.49a)$$

$$= a_o^2\sigma_u^2\, b_1^{|m|}(1-b_1^{2(k+m)+2})(1-b_1^2)^{-1} \quad m < o \qquad (3.3.49b)$$

$$= a_o^2\sigma_u^2\, b_1^{\,m}\,(1-b_1^{2k+2}).(1-b_1^2)^{-1} \quad m > o \qquad (3.3.49c)$$

Comparing eqns. (3.3.49) with eqns. (3.1.25), we see that

$$\lim_{N\to\infty} R_{\hat{y}}(k,k+m) = R_y(k,k+m) \qquad (3.3.50)$$

where $R_y(k,k+m)$ is the ensemble autocorrelation function of the output of the linear digital filter.

3.3.2 The Second Order N-bit Feedback Filter:

A second order linear digital filter follows the difference equation

$$y(kT)=a_0u(kT)+b_1y(\overline{k-1}T)+b_2\, y(\overline{k-2}T) \qquad (3.3.51)$$

The corresponding N-bit feedback coarse sampled data filter is shown in Fig. (3.3.2). The relation between the input $u(lT)$ and the output $\hat{y}(lT)$ is given by the equations:

$$z(Nl+k)=\mathrm{sgn}(a_0u(lT)+n(Nl+k)+b_1z(N\overline{l-1}+k)+b_2z(N\overline{l-2}+k)) \qquad (3.3.52a)$$

$$\hat{y}(lT)=\frac{1}{N}\sum_{k=1}^{N} z(Nl+k) \qquad (3.3.52b)$$

Since this is the coarse-sampled-data filter implementation of a linear filter of order two, therefore as stated earlier, the feedback consists of 2N elements. At $l=o$, the feedback contains $z(o)$, $z(-1)$,...$z(-2N+1)$ each of which is +1 or -1 with equal probability i.e.

$$p_{z_r}(1)=p_{z_r}(-1)=\frac{1}{2} \quad r=o,-1,-2,\ldots,-2N+1 \qquad (3.3.53)$$

The added dither $n(Nl+k)$ follows eqns. (3.3.3) and (3.3.4),

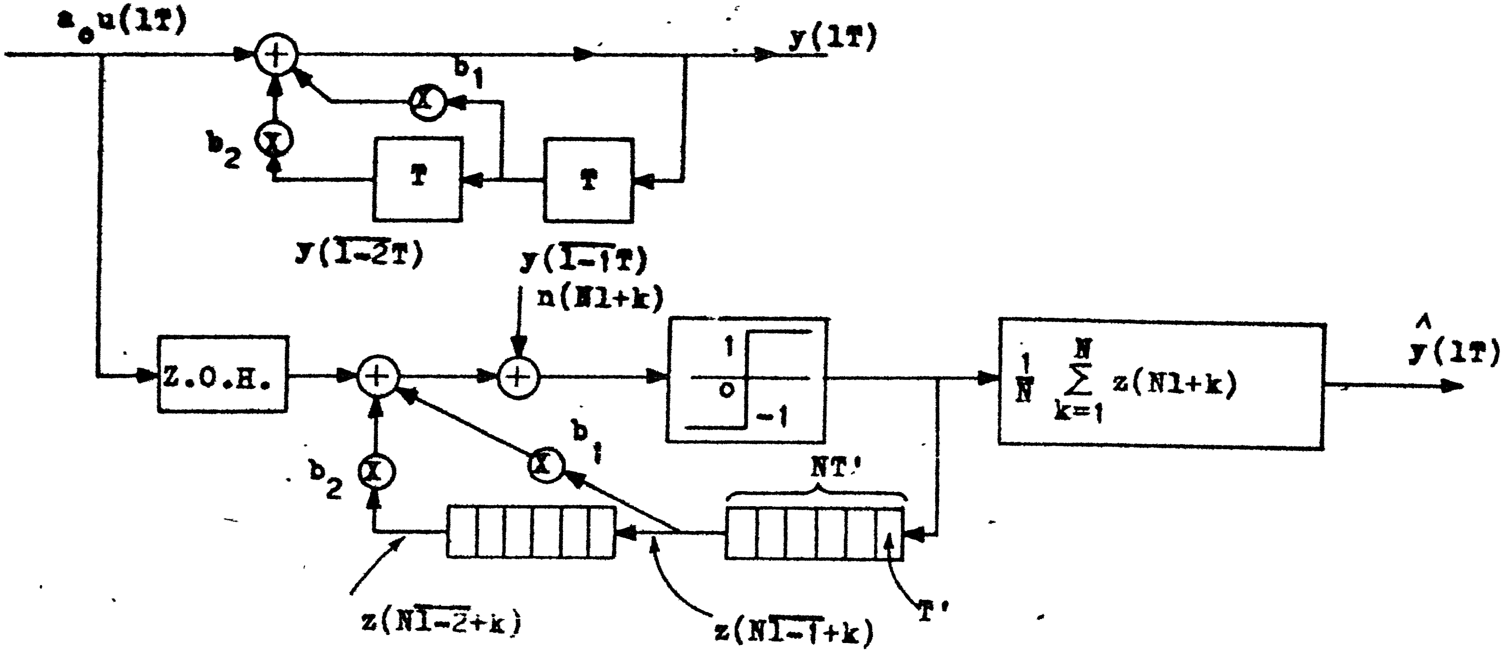


Fig. 3.3.2 Schematic of a second order linear digital filter and a second order N-bit coarse-sampled-data filter.

with M=2. For simplicity, let $b_1=o$. Hence, the non-linear filter is given by the equations

$$Z(Nl+k)=\mathrm{sgn}(a_o u(lT)+n(Nl+k)+b_2 z(N\overline{l-2}+k)) \quad (3.3.54a)$$

$$\hat{y}(lT) = \frac{1}{N} \sum_{k=1}^{N} z(Nl+k) \quad (3.3.54b)$$

(a) The deterministic input: The input u(lT) is deterministic its range obeying eqn. (3.3.3) for M=2.

(i) The probability of z(Nl+k): The probability of z(Nl+k), conditioned on $z(N\overline{l-2}+k)$ is found to be

$$p_{z_{Nl+k}|z_{N\overline{l-2}+k}}(1|z(N\overline{l-2}+k))= \frac{1+a_o u(lT)+b_2 z(N\overline{l-2}+k)}{2} \quad (3.3.55)$$

$$p_{z_{Nl+k}|z_{N\overline{l-2}+k}}(-1|z(N\overline{l-2}+k))= \frac{1-a_o u(lT)-b_2 z(N\overline{l-2}+k)}{2} \quad (3.3.56)$$

The unconditional probability of z(Nl+k) is given by a recursive equation

$$p_{z_{Nl+k}}(1) = \frac{1+a_o u(lT)-b_2}{2} + b_2\, p_{z_{N\overline{l-2}+k}}(1) \quad (3.3.57a)$$

$$= \sum_{i=o}^{l/2} \frac{1+a_o u(\overline{l-2i}T)-b_2}{2}\, b_2^i + \frac{1}{2}(.b_2^{\,l/2+1})$$

for even l (3.3.57b)

$$= \sum_{i=o}^{\frac{l-1}{2}} b_2^i . \frac{1+a_o u(\overline{l-2i}T)-b_2}{2} + \frac{1}{2} (b_2^{\frac{l+1}{2}})$$

for odd l (3.3.57c)

Similarly,

$$p_{z_{Nl+k}}(-1) = \frac{1-a_o u(lT)-b_2}{2} + b_2 \, p_{z_{N\overline{l-2}+k}}(-1) \quad (3.3.58a)$$

$$= \sum_{i=o}^{l/2} \frac{1}{2} (1-a_o u(\overline{l-2i}T)-b_2) b_2^i + \frac{1}{2}(b_2^{l+l/2})$$

for even l (3.3.58b)

$$= \sum_{i=o}^{\frac{l-1}{2}} \frac{1}{2} (1-a_o u(\overline{l-2i}T)-b_2) . b_2^i + \frac{1}{2} . b_2^{\frac{l+1}{2}}$$

for odd l (3.3.58c)

Thus $z(Nl+k)$, $k=1,2,\ldots,N$ are identically distributed and independent of 'k'.

(ii) <u>The Expected Value of $\hat{y}(lT)$</u>: Using eqn. (3.3.54b), we obtain

$$E(\hat{y}(lT)) = \frac{1}{N} \sum_{k=1}^{N} E(z(Nl+k)) \quad (3.3.59)$$

Using eqns. (3.3.57) and (3.3.58),

$$E(z(Nl+k)) = a_o \sum_{i=o}^{l/2} u(\overline{l-2i}T) . b_2^i \text{ for even } l$$

(3.3.60a)

$$=a_o \sum_{i=o}^{(l-1)/2} u(\overline{l-2iT}).b_2^i \text{ for odd } l \qquad (3.3.60b)$$

Therefore,

$$E(\hat{y}(lT)) = \frac{1}{N}(N).a_o \sum_{i=o}^{l/2} u(\overline{l-2iT}).b_2^i \text{ for even } l \qquad (3.3.61a)$$

$$= a_o \sum_{i=o}^{(l-1)/2} u(\overline{l-2iT}).\ b_2^i \quad \text{odd } l \qquad (3.3.61b)$$

$$= E(z(Nl+k)) \qquad (3.3.61c)$$

Comparing eqn. (3.3.61) with the RHS of eqn. (3.1.50) we see that the expected value of $\hat{y}(lT)$ equals the output of the linear filter $y(lT)$, when $u(lT)$ is deterministic i.e.

$$\therefore E(\hat{y}(lT)) = y(lT) \qquad (3.3.62)$$

Since $E(y(lT)) = E(z(Nl+k))$, therefore when $u(lT)$ is deterministic

$$E(z(Nl+k)) = y(lT) \qquad (3.3.63)$$

(iii) The Time Autocorrelation Function of $\hat{y}(lT)$: is defined as follows:

$$R_{\hat{y}}(m) = \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} \hat{y}(kT)\hat{y}(\overline{k+mT}) \qquad (3.3.64)$$

$R_{\hat{y}}(m)$ is a function of the added dither $n(Nl+k)$. Thus

$$E(R_{\hat{y}}(m)) = \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} E(\hat{y}(kT)\hat{y}(\overline{k+mT})) \qquad (3.3.65)$$

The inner term is given by

$$E(\hat{y}(kT)\hat{y}(\overline{k+m}T)) = E[\frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} z(Nk+i)z(N\overline{k+m}+j)] \tag{3.3.66}$$

$$= \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} E(z(Nk+i)z(N\overline{k+m}+j)) \tag{3.3.67}$$

<u>For m=o</u>

$$E(\hat{y}(k))^2 = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} E(z(Nk+i)z(Nk+j)) \tag{3.3.68}$$

$$= \frac{1}{N^2} [\sum_{i=1}^{N} E(z^2(Nk+i)) + \sum_{\substack{i=1 \\ i\neq j}}^{N} \sum_{j=1}^{N} E(z(Nk+i)z(Nk+j))] \tag{3.3.69}$$

$$= \frac{1}{N^2} [N + \sum_{\substack{i=1 \\ i\neq j}}^{N} \sum_{j=1}^{N} E[sgn(a_o u(kT)+n(Nk+i)+b_2 z(N\overline{k-2}+i)). \\ Sgn(a_o u(kT)+n(Nk+j)+b_2 z(N\overline{k-2}+j))]$$

$$= \frac{1}{N^2} [N + \sum_{\substack{i=1 \\ i\neq j}}^{N} \sum_{j=1}^{N} E\ (a_o u(kT)+b_2 z(N\overline{k-2}+i)).(a_o u(kT) \\ +b_2 z(N\overline{k-2}+j))]$$

$$= \frac{1}{N^2} [N + \sum_{\substack{i=1 \\ i\neq j}}^{N} \sum_{j=1}^{N} a_o^2 u^2(kT) + a_o b_2 u(kT) E(z(N\overline{k-2}+i) \\ +z(N\overline{k-2}+j)) + b_2^2\ E(z(N\overline{k-2}+i)z(N\overline{k-2}+j))]$$

$$= \frac{1}{N^2}\,[N + \sum_{\substack{i=1 \\ i \neq j}}^{N} \sum_{j=1}^{N} \quad a_o^2 u^2(kT) + 2a_o u(kT) . b_2 y(\overline{k-2}T) + b_2^2 .$$

$$E(z(N\overline{k-2}+i)\,z(N\overline{k-2}+j))\;] \qquad (3.3.70)$$

$$= \frac{1}{N^2}\,[N + (N^2-N)\; y^2(kT)] \qquad (3.3.71)$$

Therefore,

$$E(R_{\hat{y}}(o) = \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} \frac{1}{N^2}\,[N + (N^2-N)y^2(kT)] \qquad (3.3.72)$$

In the limit, as $N \to \infty$, $E(R_{\hat{y}}(m))$ tends to

$$\lim_{N\to\infty} E(R_{\hat{y}}(o)) = \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} y^2(kT) \qquad (3.3.73)$$

But the RHS of eqn. (3.3.73) is the time autocorrelation $R_y(m)$ of $y(kT)$ at $m=o$

$$\therefore \lim_{N\to\infty} E(R_{\hat{y}}(o)) = R_y(o) \qquad (3.3.74)$$

For $m \neq o$:-

$$E(\hat{y}(kT)\hat{y}(\overline{k+m}T)) = E[\frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} Z(Nk+i)\,z(N\overline{k+m}+j)]$$

$$= \frac{1}{N^2}\,[\sum_{i=1}^{N} E(z(Nk+i)\,z(N\overline{k+m}+i)) + (N^2-N)E(z(Nk+i)$$

$$Z(Nk+m+j))_{\;i\neq j}] \qquad (3.3.75)$$

[illegible] $+y(kT) . \sum_{q=o}^{m/2} b_2^q\, u(\overline{k+m-2q}T))$

$+(N^2-N)y(kT) . y(\overline{k+m}T)]$ $m > o$ and even

(3.3.76a)

$$= \frac{1}{N^2}\,[N(b_1^{|m|/2} + y(\overline{k+m}T) \sum_{q=o}^{m/2} b_2^q\, u(\overline{k-2}qT)) + (N^2-N)\cdot$$

$$y(kT)y(\overline{k+m}T)] \quad m<o \text{ and even} \qquad (3.3.76b)$$

If m is odd, $z(Nk+i)$, $z(Nk+m+i)$ are always independent, for all i,j, therefore

$$E(\hat{y}(kT)\hat{y}(\overline{k+m}T)) = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} E(z(Nk+i)).E(z(N\overline{k+m}+j))$$

(3.3.77a)

$$= \frac{1}{N^2} \cdot N^2 . y(kT) . y(\overline{k+m}T) = y(kT)y(\overline{k+m}T)$$

(3.3.77b)

Using eqns. (3.3.76b) and (3.3.76), we see that, for all m, in the limit, as $N \to \infty$

$$\lim_{N\to\infty} E(\hat{y}(kT)\hat{y}(k+mT) = y(kT)y(\overline{k+m}T) \quad \text{all } m \qquad (3.3.78)$$

$$\therefore \lim_{N\to\infty} E(R_y^*(m)) = \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=o}^{K} y(kT)y(\overline{k+m}T) \quad \text{all } m$$

$$= R_y(m) \qquad (3.3.79)$$

where $R_y(m)$ is the time autocor... ...ion function of $y(kT)$.

(b) The random input: The input $u(lT)$ is random, its range obeying eqn. (3.3.3) for M=2.

(i) The Probability of $z(Nl+k)$: The probability of $z(Nl+k)$, conditioned on both $u(lT)$ and $z(N\overline{l-2}+k)$ is found to be

$$p_{z_{Nl+k}\,|\,z_{N\overline{l-2}+k},\,u_{lT}}\left(1\,\middle|\,z(N\overline{l-2}+k),u(lT)\right)=\frac{1+a_o u(lT)+b_2 z(N\overline{l-2}+k)}{2} \tag{3.3.80}$$

$$p_{z_{Nl+k}\,|\,z_{N\overline{l-2}+k},\,u_{lT}}\left(-1\,\middle|\,z(N\overline{l-2}+k),u(lT)\right)=\frac{1-a_o u(lT)-b_2 z(N\overline{l-2}+k)}{2} \tag{3.3.81}$$

$$\therefore\ p\left(z(Nl+k)\,\middle|\,z(N\overline{l-2}\cdot k)\right)=\int_u p(\ldots \text{[illegible]} (N\overline{l-2}+k),\,u(lT)) \cdot p(u(lT)).du(lT) \tag{3.3.82}$$

Therefore,

$$p_{z_{Nl+k}\,|\,z_{N\overline{l-2}+k}}\left(1\,\middle|\,z(N\overline{l-2}+k)\right)=\frac{1+a_o\eta_u+b_2 z(N\overline{l-2}+k)}{2} \tag{3.3.83}$$

$$p_{z_{Nl+k}\,|\,z_{N\overline{l-2}+k}}\left(-1\,\middle|\,z(N\overline{l-2}+k)\right)=\frac{1-a_o\eta_u-b_2 z(N\overline{l-2}+k)}{2} \tag{3.3.84}$$

The unconditional probability of $z(Nl+k)$ is given by the recursive equation

$$p_{z_{Nl+k}}(1) = \frac{1+a_o\eta_u-b_2}{2} + b_2 p_{z_{N\overline{l-2}+k}}(1) \tag{3.3.85a}$$

$$= \frac{1+a_o\eta_u-b_2}{2} \sum_{i=o}^{l/2} b_2^i + \frac{1}{2} . b_2^{l/2+1} \quad \text{for even } l \tag{3.3.85b}$$

$$= \frac{1+a_o\eta_u-b_2}{2} \sum_{i=o}^{(l-1)/2} b_2^i + \frac{1}{2} . b_2^{(l+1)/2} \quad \text{for odd } l \tag{3.3.85c}$$

Similarly,

$$p_{z_{Nl+k}}(-1) = \frac{1-a_o\eta_u-b_2}{2} + b_2\, p_{z_{N\overline{l-2}+k}}(-1) \tag{3.3.86a}$$

$$= \frac{1-a_o\eta_u-b_2}{2} \sum_{i=o}^{l/2} b_2^i + b_2^{l/2+1} \quad \text{even } l \tag{3.3.86b}$$

$$= \frac{1-a_o\eta_u-b_2}{2} \sum_{i=o}^{(l-1)/2} b_2^i + \frac{b_2^{(l+1)/2}}{2} \quad \text{odd } l \tag{3.3.86c}$$

Hence $z(Nl+k), k=1,2,\ldots,N$ are identically distributed and independent of 'k'.

(ii) The Expected Value of $\hat{y}(lT)$:

$$E(\hat{y}(lT)) = \frac{1}{N} \sum_{k=1}^{N} E(z(Nl+k)) \tag{3.3.87}$$

$$E(z(Nl+k)) = a_o\eta_u \sum_{i=o}^{l/2} b_2^i \quad \text{for even } l \tag{3.3.88a}$$

$$= a_o\eta_u \sum_{k=o}^{(l-1)/2} b_2^i \quad \text{for odd } l \tag{3.3.88b}$$

Hence

$$E(\hat{y}(lT)) = a_o\eta_u \sum_{i=o}^{l/2} b_2^i \quad \text{for even } l \tag{3.3.89a}$$

$$= a_o\eta_u \sum_{i=o}^{(l-1)/2} b_2^i \quad \text{for odd } l \tag{3.3.89b}$$

Comparing eqns. (3.3.89) and (3.3.88) with (3.1.50), we see that $E(\hat{y}(lT))$ equals the expected value of $y(lT)$, and that $E(z(Nl+k))$ also equals $E(y(lT))$ i.e.

$$E(\hat{y}(lT))=E(z(Nl+k))=E(y(lT)) \quad k=1,2,\ldots,N \qquad (3.3.90)$$

(iii) The Ensemble Autocorrelation of $\hat{y}(lT)$: is defined by

$$R_{\hat{y}}(k,k+m)=E[\hat{y}(kT)\hat{y}(\overline{k+m}T)]$$

$$=E[\frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} z(Nk+i)z(N\overline{k+m}+j)] \qquad (3.3.91)$$

$$R_{\hat{y}}(k,k+m)\Big|_{m=o} = \frac{1}{N^2} [N+ \sum_{\substack{i=1\\ i\neq j}}^{N} \sum_{j=1}^{N} E[z(Nk+i)z(Nk+j)]$$

$$= \frac{1}{N^2} [N+(N^2-N) \; E(sgn(a_o u(kT)+n(Nk+i)+b_2 z(N\overline{k-2}+i)).$$

$$sgn(a_o u(kT)+n(Nk+j)+b_2$$

$$z(N\overline{k-2}+j)) \;]_{i\neq j}$$

$$= \frac{1}{N^2} [N+(N^2-N) \quad a_o^2 E(u^2(kT))+2a_o E(u(kT)).$$

$$b_1 E(y(\overline{k-2}T))+b_2^2 \; E(z(N\overline{k-2}+i)z(N\overline{k-2}+j))]_{i\neq j}$$

$$= \frac{1}{N^2} [N+(N^2-N)E(y^2(kT))] \qquad (3.3.92)$$

In the limit as $N \to \infty$

$$\lim_{N\to\infty} R_{\hat{y}}(k,k+m)\Big|_{m=o} = E(y^2(kT))$$

$$= R_y(k,k) \qquad (3.3.93)$$

When m is odd, and $z(Nk+i)$ and $y(kT)$ are given by eqns. (3.3.54a) and (3.1.50) respectivel,, then $z(Nk+i), z(N\overline{k+m}+j)$ are mutually independent for all i,j. Similarly $y(kT)$, $y(\overline{k+m}T)$ are independent. Therefore using (3.3.91),

$$R_{\hat{y}}(k,k+m) = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} E(z(Nk+i)).E(z(N\overline{k+m}+j)) \qquad (3.3.94)$$

$$= E(y(kT)).E(y(\overline{k+m}T))$$

$$= o \quad (\because \eta_u = o) \quad \text{for odd } m \qquad (3.3.95)$$

When m is even, and k=2n or k=2n+1 and k+m=2r or k+m=2r+1

$$R_{\hat{y}}(k,k+m) = \frac{1}{N^2} \Big[\sum_{i=1}^{N} E(z(Nk+i)z(N\overline{k+m}+i)) + \sum_{\substack{i=1\\ i\neq j}}^{N} \sum_{j=1}^{N} E(z(Nk+i)z(N\overline{k+m}+j))\Big] \qquad (3.3.96)$$

$$= \frac{1}{N^2} \Big[N.b_2^{|m|/2} + b_2^{|m|/2} \sum_{\substack{i=1\\ i\neq j}}^{N} \sum_{j=1}^{N} E(z(Nk+i).z(Nk+j))\Big] \qquad (3.3.97)$$

$$R_{\hat{y}}(k,k+m)\Big|_{m>o} = \frac{1}{N^2} [N.b_2^{m/2} + b_2^{\;/2}.(N^2-N)E(y^2(kT))] \qquad (3.3.98a)$$

Substituting from (3.1.54a) in (3.3.98a), we obtain

$$R_{\hat{y}}(k,k+m)\Big|_{m>o} = \frac{1}{N^2} \Big[N.b_2^{m/2} + b_2^{m/2}(N^2-N).a_{\eta}^2\sigma_u^2 . \frac{1-b_2^{2n+2}}{1-b_2^2}\Big] \qquad (3.3.98b)$$

$$R_{\hat{y}}(k,k+m)\Big|_{m<0} = \frac{1}{N^2}\,[N.b_2^{|m|/2} + b_2^{|m|/2}.\ \sum_{\substack{i=1}}^{N}\ \sum_{\substack{j=1\\ i\neq j}}^{N}$$

$$E(z(N\overline{k+m}+i)\,z(N\overline{k+m}+j))] \qquad (3.3.99)$$

$$R_{\hat{y}}(k,k+m)\Big|_{m<0} = \frac{1}{N^2}\,[N.b_2^{|m|/2} + (N^2-N).b_2^{|m|/2}.E(y^2(\overline{k+m}T))]$$

(3.3.100a)

Substituting from (3.1.54a) in (3.3.100a), we obtain,

$$R_{\hat{y}}(k,k+m)\Big|_{m<0} = \frac{1}{N^2}\,[N.b_2^{|m|/2} + (N^2-N).b_2^{|m|/2}\ .\ a_o\sigma_u^2.\ \frac{1-b_2^{2r+2}}{1-b_2^2}\ ]$$

(3.3.100b)

Using eqns. (3.3.95), (3.3.98b), (3.3.100b), in the limit, as $N \rightarrow \infty$

$$\lim_{N\to\infty} R_{\hat{y}}(k,k+m) = R_y(k,k+m) \qquad \text{all } m.$$

SUMMARY:

In this chapter the coarse-sampled-data filter and the N-bit feedback coarse-sampled-data filter have been investigated. It is found that the performance of the coarse-sampled-data filter is not close to the performance of the output of the corresponding linear digital filter whereas

for both the cases of the deterministic and the random inputs, the expected value of the output $\hat{y}$ of the N-bit feed back sampled data filter is equal to the expected value of the output y of the linear filter. It is also shown that as N increases, i.e. as the dither rate T' increases for a fixed input sampling rate T(T'=T/N), the auto-correlation function of $\hat{y}$, the output of the N-bit feedback sampled-data-filter, asymptotically approaches the auto-correlation function of y the output of the linear digital filter i.e.

$$E(\hat{y}) = E(y)$$

and $$R_{\hat{y}}(k,k+m) = R_y(k,k+m).$$

At steady state, i.e. with k tending to infinity, y(lT) becomes stationary i.e. $\lim_{k\to\infty} R_y(k,k+m)=R_y(m)$, therefore when the N-bit feedback filter is allowed to run for some time, its output must become stationary, i.e. when the non-linear filter reaches steady state, the power spectrum of its output may be defined. Because the autocorrelation functions of the outputs of the linear and nonlinear filters are equal in steady state, therefore the power spectra must be equal also i.e.

$$S_{\hat{y}}(\omega) = S_y(\omega)$$

When a comparison is made of the hardware requirements for the coarse-sampled-data filter and the N-bit feedback

that at time $t=Nl+k$, $k=1,..N$, the values $z(Ns+N)$, $s=l-1$, $l-2$, $l-M$ are stored in the feedback register, i.e. the feedback register length must be M. It can be shown that the expectation of the output of this filter equals that of the output of the linear digital filter. However, the autocorrelation of its output is larger than that of the output of the N-bit feedback filter, while being lesser than that of the output of the coarse-sampled-data filter. Here, a tradeoff between the cost of the filter and the degree of the accuracy of the output of the filter, must be made, in deciding on the nonlinear filter. A study by simulation has been made only of the N bit feedback sampled-data filter, however.

CHAPTER 4

RESULTS OF SIMULATION OF N-BIT FEEDBACK COARSE-SAMPLED-DATA FILTERS

In Chapter 3, a study was made of what was called the N-bit feedback coarse-sampled-data filter which was developed to replace a linear digital filter. The advantage was a simplification in hardware. In theory (Chap. 3.3) it was found that the output of this nonlinear filter could be made to approximate as closely as desired the output of the corresponding linear digital filter in the correlation-invariance sense.

This chapter outlines the study made of the performance of the nonlinear filter by simulating it on the computer DEC-10, which is a 36-bit wordlength machine. Sec. 4.1 discusses the program written for the DEC-10 to simulate the nonlinear filter. Sec. 4.2 discusses the performance of 1st and 2nd order nonlinear filters as obtained by simulating them on the machine.

4.1 Simulation in Fortran-10:

A program NLF (Non-Linear Filter) was written in the computer language FORTRAN-10 to simulate the nonlinear N-bit feed-back coarse-sampled-data filter on

the DEC-10 computer. A listing of the program is provided in Appendix A. The variable names used for some pertinent quantities, input, output and intermediate, are explained as follows:

The linear difference equation from which the nonlinear filter is derived is given by

$$y(lT)= \sum_{i=1}^{RU} a_i\ u(\overline{l-i+1}T)+ \sum_{j=1}^{RY} b_j\ y(\overline{l-j}T) \tag{4.1.1}$$

Consequently, the nonlinear filter is given by the equations

$$Zhat = \sum_{i=1}^{RU} a_i u(\overline{l-i+1}T)+ n(Nl+k)+ \sum_{j=1}^{RY} b_j z(N\overline{l-j}+k) \tag{4.1.2}$$

$$z(Nl+k) = Zhat/|Zhat| \tag{4.1.3}$$

$$Yhat(lT) = \left(\sum_{k=1}^{N} z(Nl+k)\right) LIMN/N \tag{4.1.4}$$

Here $n(Nl+k)$ is given by

$$n(Nl+k)=(LIMN).(RFB(Nl+k)-0.5).2.0 \tag{4.1.5}$$

where RFB(Nl+k) is a pseudo-random number lying in the range (0,1) and generated by the computer library program GGUB. The factor '2.0' normalizes the value (RFB(Nl+k)-0.5) to the range (-1,1). LIMN is the magnitude of the

maximum excursion of $n(Nl+k)$ either above or below zero. In terms of the filter coefficients and the input $u(lT)$

$$LIMN \geqslant \left[\sum_{i=1}^{RU} a_i \, u(\overline{l-i+1}T) \right]_{max} + \sum_{j=1}^{RY} \left| b_j \right| \qquad (4.1.6a)$$

$$-LIMN \leqslant \left[\sum_{i=1}^{RU} a_i \, u(\overline{l-i+1}T) \right]_{min} - \sum_{j=1}^{RY} \left| b_j \right| \qquad (4.1.6b)$$

Other variables used are T,N,MU and M. They are explained thus:

T : is the input/output sampling period.

N : relates the period T' of the dither n(Nl+k) to the period T of the input, thus T'=T/N. If the order of the linear digital filter is RY, the number of elements in the feedback register is (RY)(N).

MU: (MU).T is the time allowed to elapse after which the filter is assumed to be in steady state i.e. the output of the nonlinear filter is assumed to be stationary.

M : At time (MU).T, when the filter is assumed to be well into steady state, the previous M values of say, the output i.e. Yhat($\overline{MU-M+1}T$), ..., Yhat(MU) are used to obtain the power spectrum of the variable, which in this case is Yhat.

Before the first value of u(lT) is input to the system, the feedback (z(j), j=o,-1, -N(RY-1)+1) is initialized. A pseudo-random number YY, generated using the external defined function RNDY1(X) on the computer, is used (YY=RNDY1(X). RNDY1(X), and hence YY, lies in the range (0,1). To satisfy the condition

$$p_{z_j}(1) = p_{z_j}(-1) = \frac{1}{2} \quad (\quad j= 0,-1,\ldots,-N(RY-1)+1) \tag{4.1.7}$$

z(j) is initialised to +1(-1) as YY is greater than or equal to (less than) 0.5, or vice versa. The filter may now be allowed to run, generating an output value Yhat(lT) for every input value u(lT). It is seen that, to reduce the possibility of any correlation between the dithering noise n(Nl+k) and the initial values of the feedback z_j, j=0,-1,...,-N(RY-1)+1 , their generators in the system must be two physically different, independent random number generaters, as GGUB and RNDY1 are.

The M-point power spectrum of a particular variable is obtained by using the subroutine MEM, which uses the Maximum Entropy Method for the derivation of the power spectrum of a variable from corrupted data.

4.2 Simulation Results:

The computer program used to test the performence of the nonlinear N-bit feedback sampled-data filter has been described in the previous section. The program was used to simulate first and second order nonlinear filters.

<u>The First Order Filter</u>: follows the linear difference equation

$$y(kT) = u(kT) + \alpha y(\overline{k-1}T) \qquad (4.2.1)$$

where, to ensure the stability of the filter, $|\alpha| < 1$. The performance of the first order nonlinear filter was tested for various inputs.

(a) Input Zero: When the input to the nonlinear filter is zero, the output y(lT) is given by the equations

$$z(Nl+k) = \mathrm{Sgn}(n(Nl+k) + \sum_{i=1}^{RY} b_i \; z(\overline{Nl-i}+k)) \qquad (4.2.2)$$

$$\hat{y}(lT) = \frac{1}{N} \sum_{k=1}^{N} z(Nl+k) \qquad (4.2.3)$$

The output therefore shall be the natural or the initial condition response of the nonlinear filter. It is zero mean and converges to zero with time because circulation through the filter causes the feedback to be weighted by α^l which is a decaying term. This is illustrated in Fig. 4.1, where the output was obtained by setting u(nT)

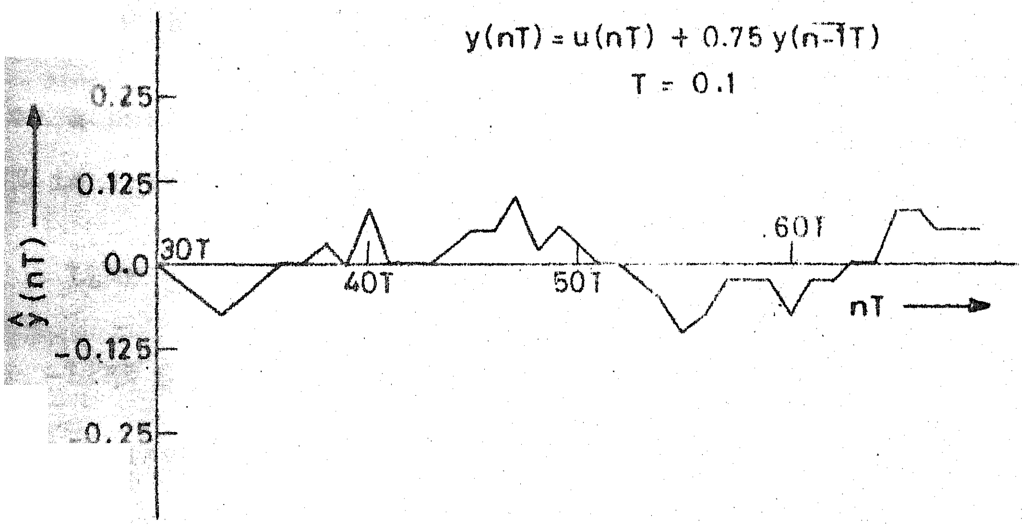


Fig. 4.1 Initial condition response of first order N-bit [illegible] sampled-data filter

to zero and allowing the filter to run. The output y(nT) is seen to be zeromean, and close to zero.

(b) Step Input: When the input u(nT) to the linear filter is a step function, the output y(nT) must ultimately reach a constant value given by

$$y(nT) = \frac{u(o)}{1-\alpha} \tag{4.2.4}$$

Because $E(\hat{y}(nT)) = E(y(nT))$

Therefore $E(\hat{y}(nT) = \frac{u(o)}{1-\alpha}$ (4.2.5)

For a particular u(nT) and α, the range of the dither LIMN is determined from

$$\begin{aligned} LIMN &\geqslant u_{max} + |\alpha| \\ -LIMN &\leqslant u_{min} - |\alpha| \end{aligned} \tag{4.2.6}$$

In this case $u_{max} = u_{min} = u_o$

$$\begin{aligned} LIMN &\geqslant u_o + |\alpha| \\ -LIMN &\leqslant u_o - |\alpha| \end{aligned} \tag{4.2.7}$$

$$\therefore \; LIMN = \max \left[\left| u_o + |\alpha| \right| , \left| u_o - |\alpha| \right| \right] \tag{4.2.8}$$

The maximum value that $\hat{y}(nT)$ may attain is

$$\hat{y}(nT)_{max} = \frac{1}{N} (\sum_{k=1}^{N} Z(Nn+k)_{max} \, LIMN$$

$$= \frac{1}{N} \cdot N \; LIMN = LIMN \tag{4.2.9}$$

Therefore if LIMN $< \frac{u_o}{1-\alpha}$, the output of the nonlinear filter shall saturate at $\pm$LIMN. This implies that the range of the input u(nT) is constrained by the inequality

$$\max [\, u_o + \alpha \;,\; u_o - \alpha \,] > \frac{u_o}{1-\alpha} \qquad (4.2.10)$$

This is illustrated in Fig. 4.2 for which α=0.75. The output corresponding to an input = 0.5 to the linear digital filter i.e. 2.0 LIMN=1.25. The nonlinear filter output saturates at 1.25.

(c) The sinusoidal Input: The performance of the first order N-bit feedback sampled-data filter is evaluated when the input is sinusoidal of the form

$$u(kT) = \cos\left(\frac{2\pi k}{A}\right) \qquad (4.2.11)$$

the period of the input being AT seconds. Eqn. (3.3.10) for the expected value of the output of a linear filter is

$$E(y(lT)) = \sum_{i=0}^{l} b_1^i \, u(\overline{l-i}T) \qquad (4.2.12)$$

The output y(lT) is therefore the weighted sum of sinusoidal terms, each of period AT secs. Hence y(lT) must be periodic with period AT secs.

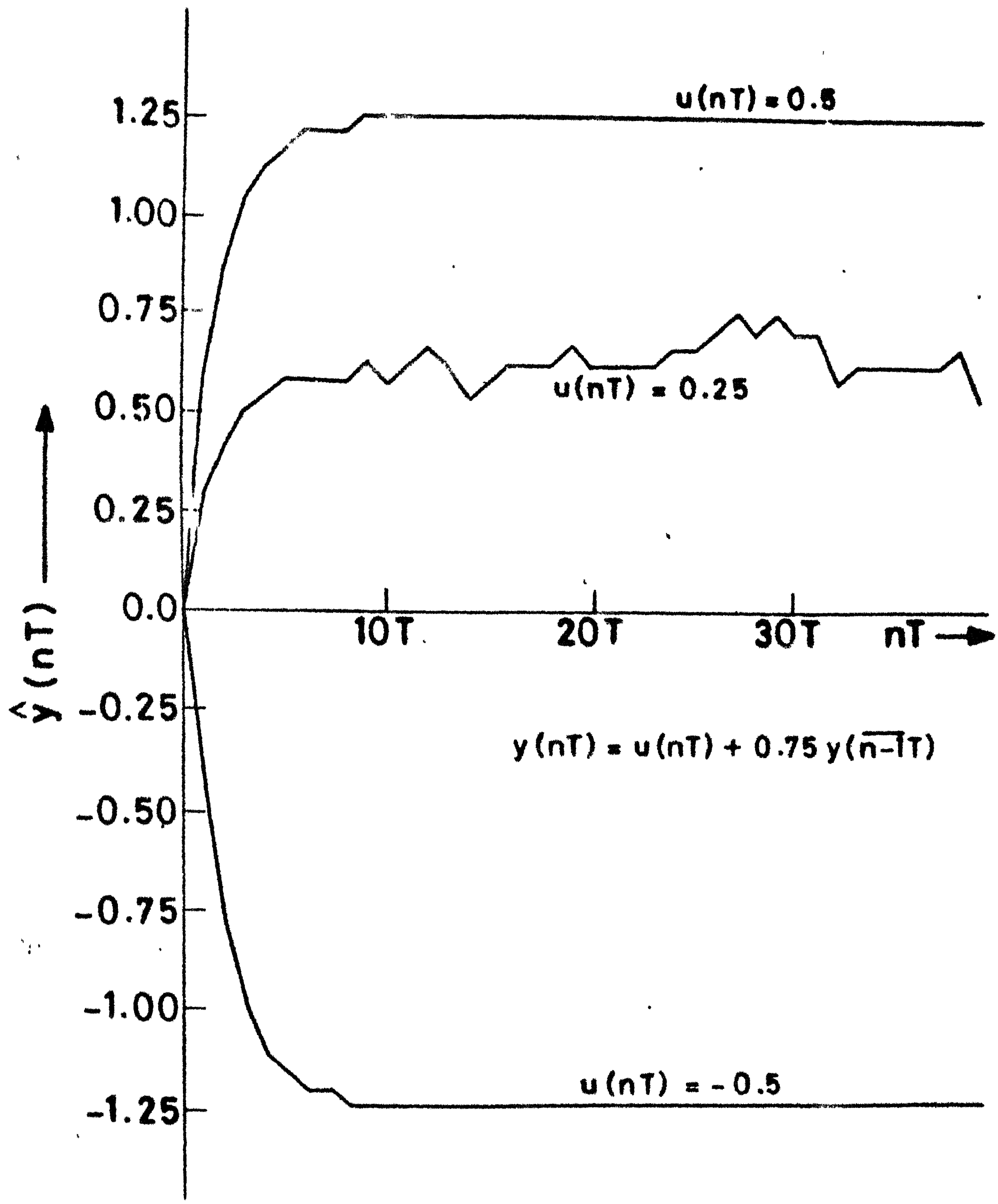


Fig. 4.2 First order filter step response
LIMN = 1.25 , N = 512

The output of the N-bit feedback sampled-data filter, corresponding to the linear difference equation

$$y(kT) = u(kT) + 0.75\, y(\overline{k-1}T) \qquad (4.2.13)$$

where the period of the input is 16T, is depicted in Fig. 4.3(a). The output $\hat{y}(lT)$ is indeed seen to be sinusoidal with period 16T. Fig. 4.3(b) shows the power spectra of the outputs of the linear and non-linear filters and the power spectrum of the sinusoidal input, which has a period of 16T secs. The figure (4.3b) shows a peak for the same frequency 1/16T for the outputs of the linear and the N-bit feedback sampled-data filters and for the input which is sinusoidal. There is a finite dispersion about the peak frequency in each case. This is due to the fact that a finite number of points (M=80) were used to determine the power spectrum in each case.

(d) White Noise Input: Sec. (3.3.1) discussed in theory the performance of a first order nonlinear digital filter, to a white noise input. Here we discuss the performance of first order nonlinear filters corresponding to linear digital filters given by Eqn. (4.2.1) when the input is white noise.

When simulating a first order nonlinear filter, for a particular run of the simulator, the parameters MU and N,

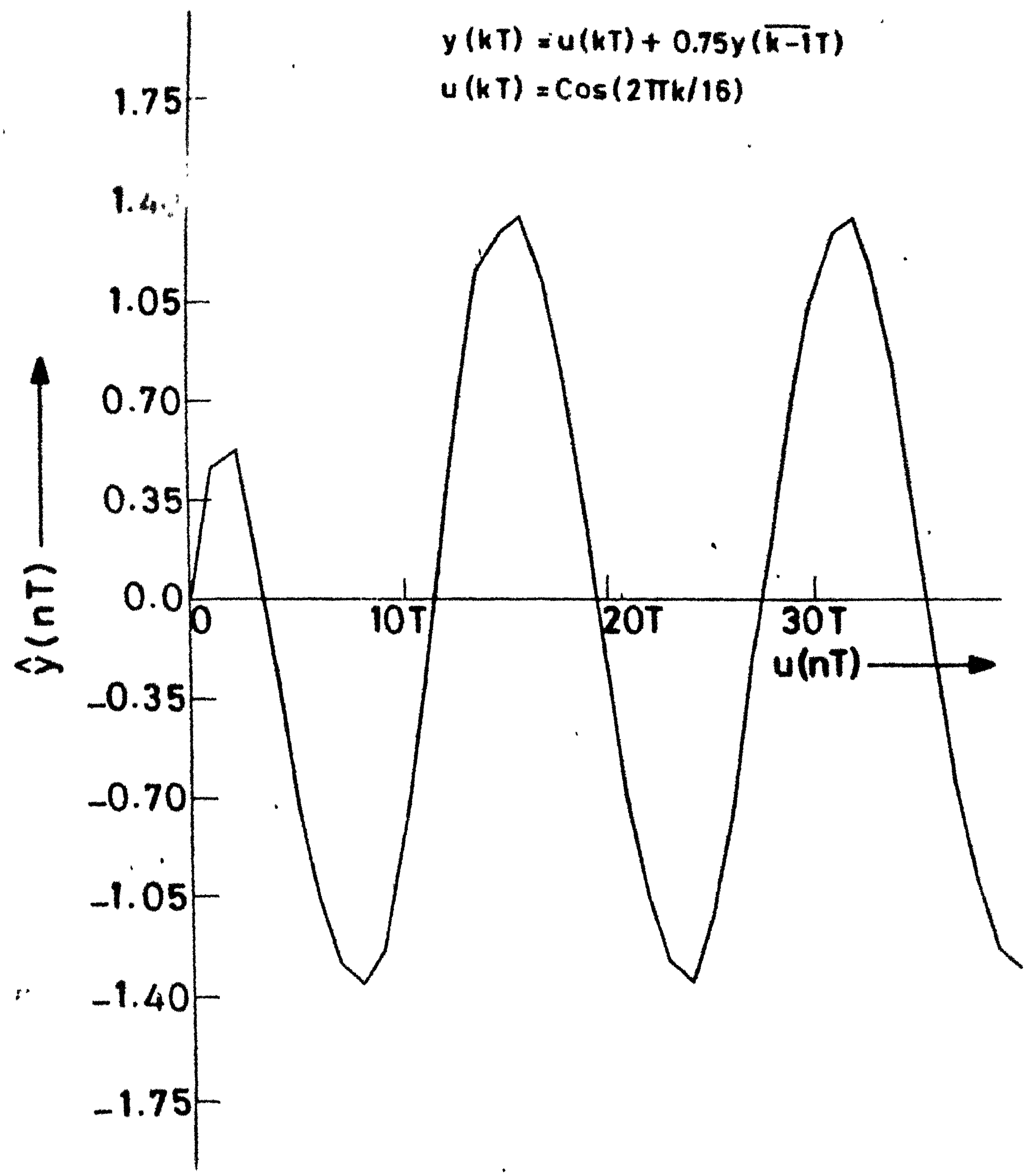


Fig.4.3(a) Response of first order N-bit sampled-data filter to sinusoidal input.

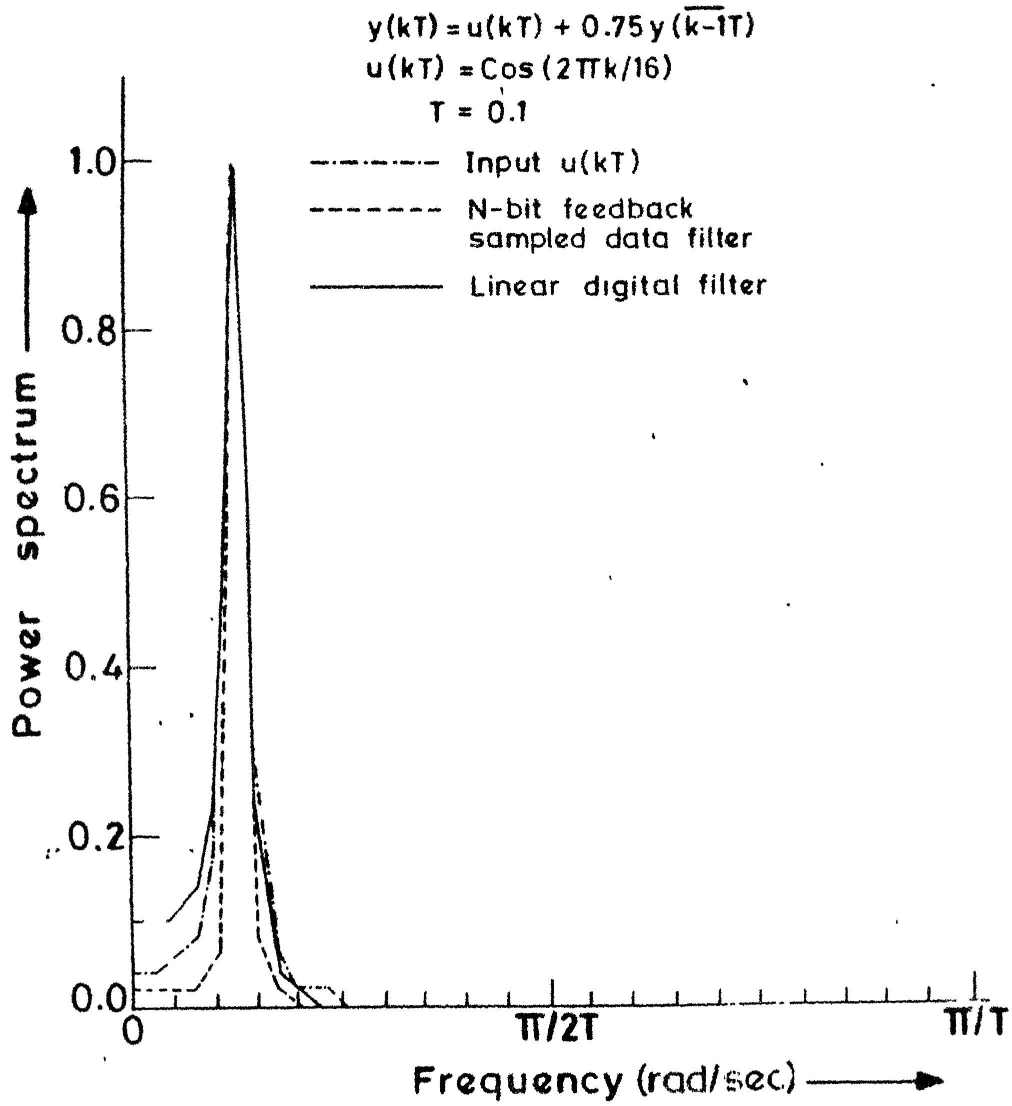


Fig.4.3(b) First order filters, output power spectra for sinusoidal input.

which have been previously defined in Sec. 4.1, have particular values.

Fig. (4.4a), (4.4b), (4.4c) compare the power spectra of the outputs of nonlinear N-bit feedback - coarse-sampled-data filters with those of the corresponding linear digital filters. For α=0.25, (0.5), (0.75) the values of MU and N that give the closest power spectrum matching are MU=512, (1024), (512) and N=512,(512),(1024).

The parameter N relates the dither period T' to the input sampled period T (T'=T/N). Increasing (decreasing) N implies that the dither period T' decreases (increases) for a fixed input sampling rate T. It is seen that by obtaining the output power spectra for a nonlinear filter, for a range of values of N there is a particular values of N that yields the most close matching of the o/p power spectrum with that of the o/p of the linear digital filter. An intuitive explanation may be given. In theory an infinite time i.e. N= ∞ is required for perfect matching of the power spectra of the outputs of the nonlinear filter and the linear digital filter. However, this is not practicable. Hence for low values of N, there shall be a finite error between the spectra. As N increases, this error may decrease. An inherent feature in any finite wordlength machine is the introduction of round off/truncation

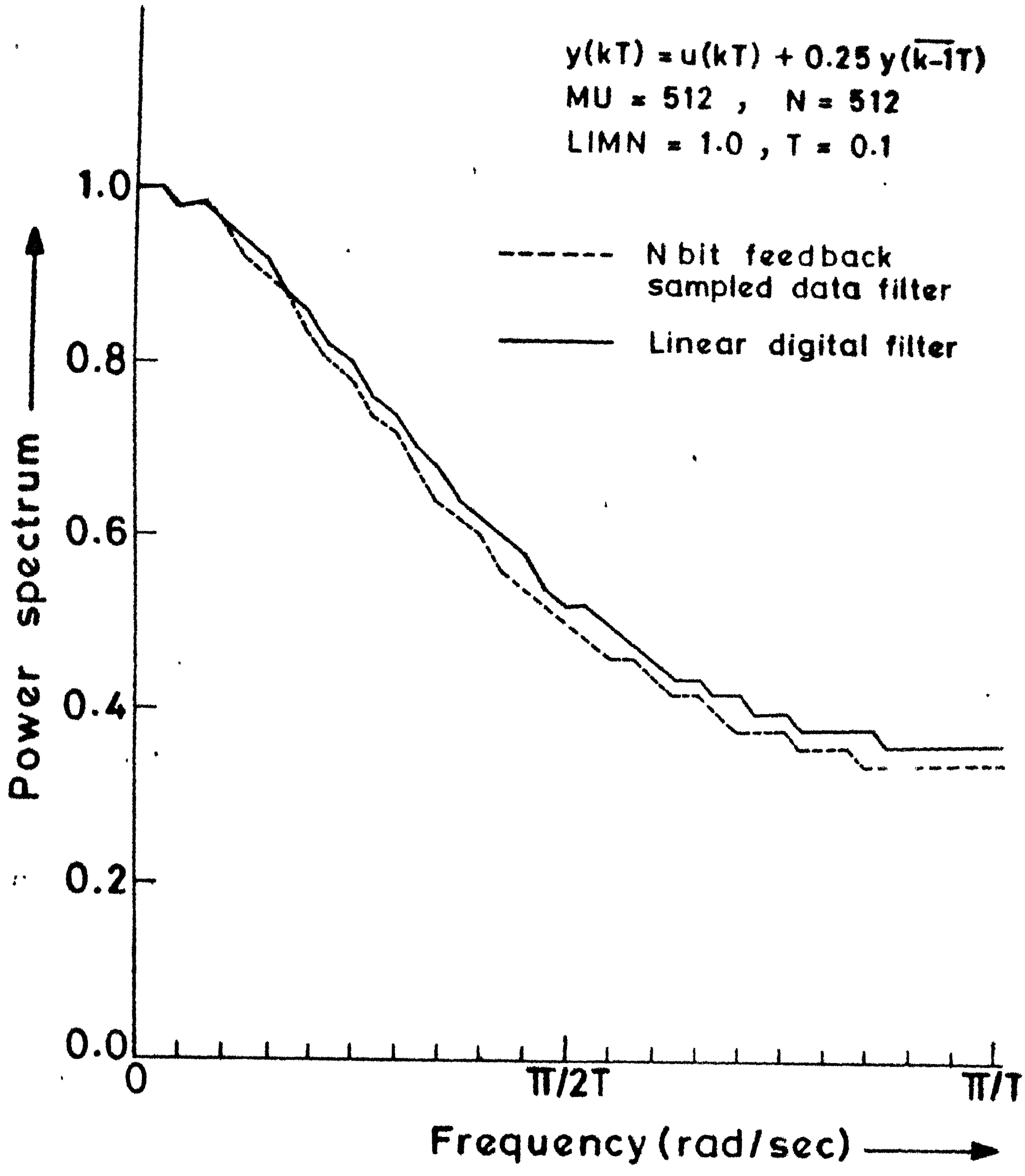


Fig. 4.4(a) First order filters, output power spectra.

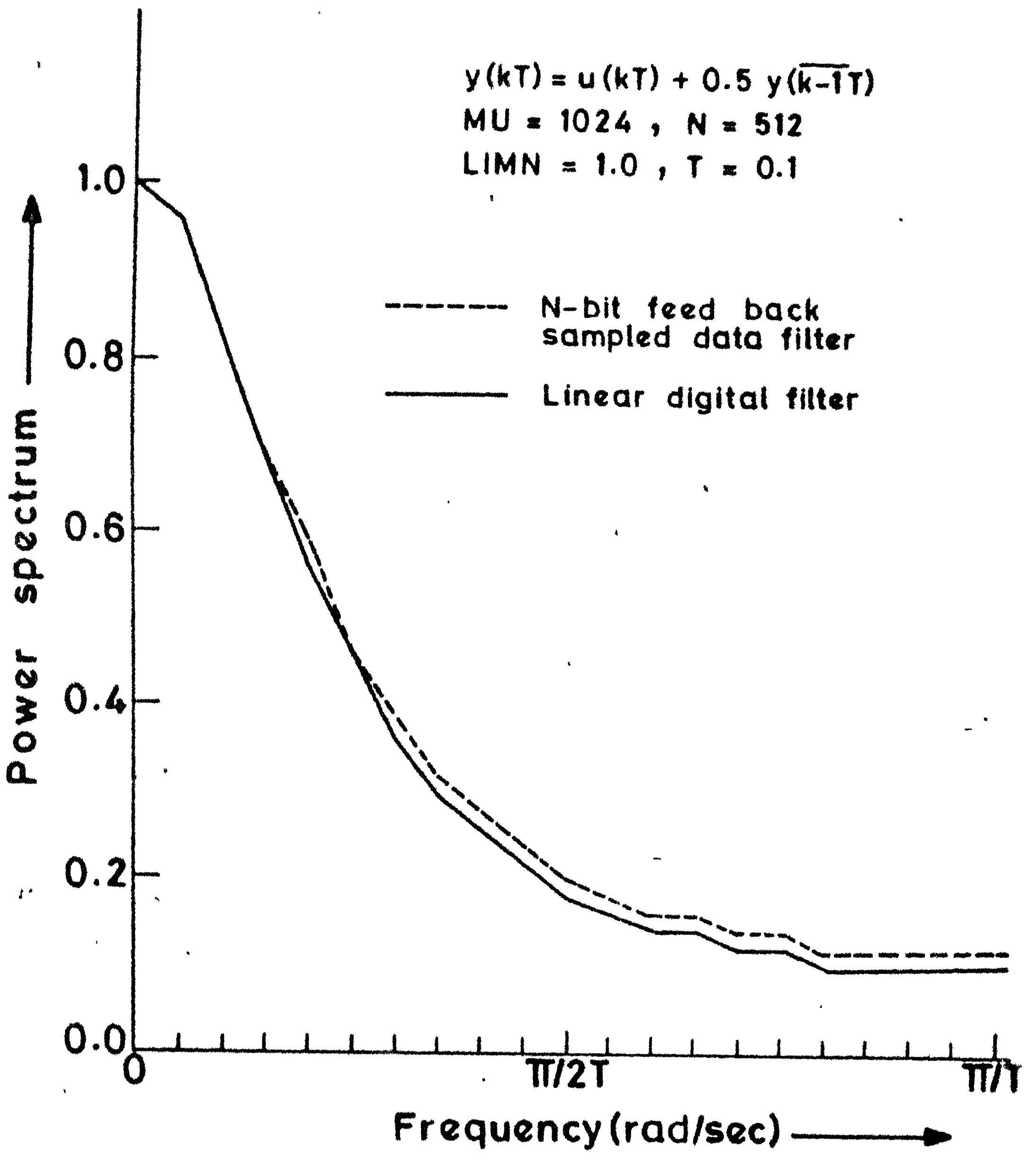


Fig.4.4(b) First order filters, output power spectra.

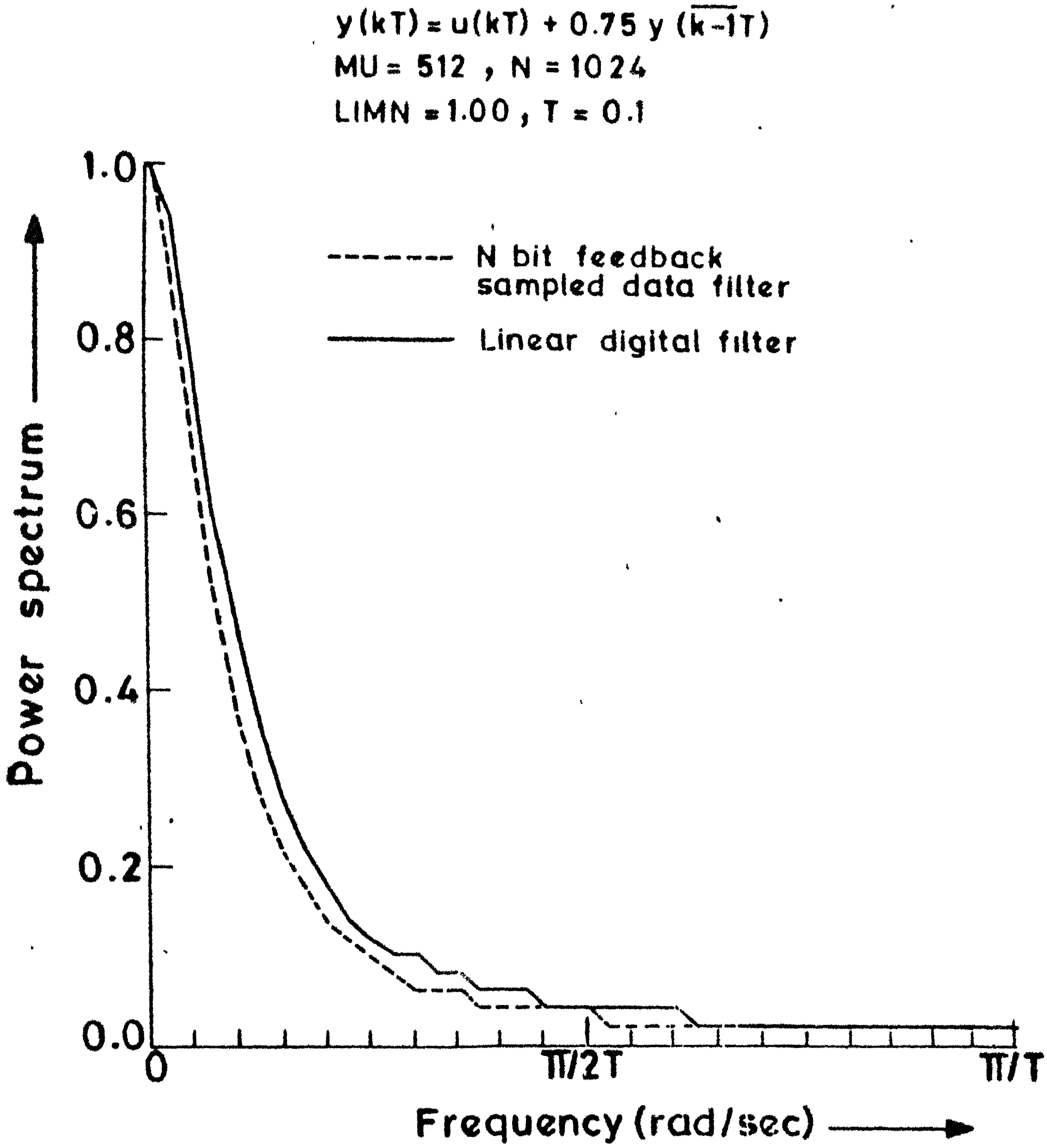


Fig. 4.4(c) First order filters, output power spectra.

noise into the filter. As N increases boundlessly, the cumulative effect of the cut-off/truncation noise may increase and cause the error between the spectra of the linear and the nonlinear filters to increase. However, this explanation remains an intuitive one, as no study, either theoretical or by simulation has been made to back it.

The initial condition response dies down as α^1. Hence MU must be large enough for the initial condition response to be negligible. Therefore MU must be greater as $|\alpha| \to 1$. However the simulation results show optimal values of MU are MU=512 for α = 0.25 and 0.75 and MU=1024 for α=0.5 for each of which $\alpha^{MU} \simeq 0$.

<u>Second Order Filters</u>: Figs 4.5(a),(b),(c) depict the spectra for the linear digital filters corresponding to

$$y(kT) = u(k)+\alpha_1 y(\overline{k-1}T) + \alpha_2 y(\overline{k-2}T) \qquad (4.2.14)$$

and the N-bit feedback sampled-data filters derived from them, both for white noise input.

The simulation program is run to enable one to study the effect of MU and N on the performance of the filter. The behaviour of the nonlinear filters' performance

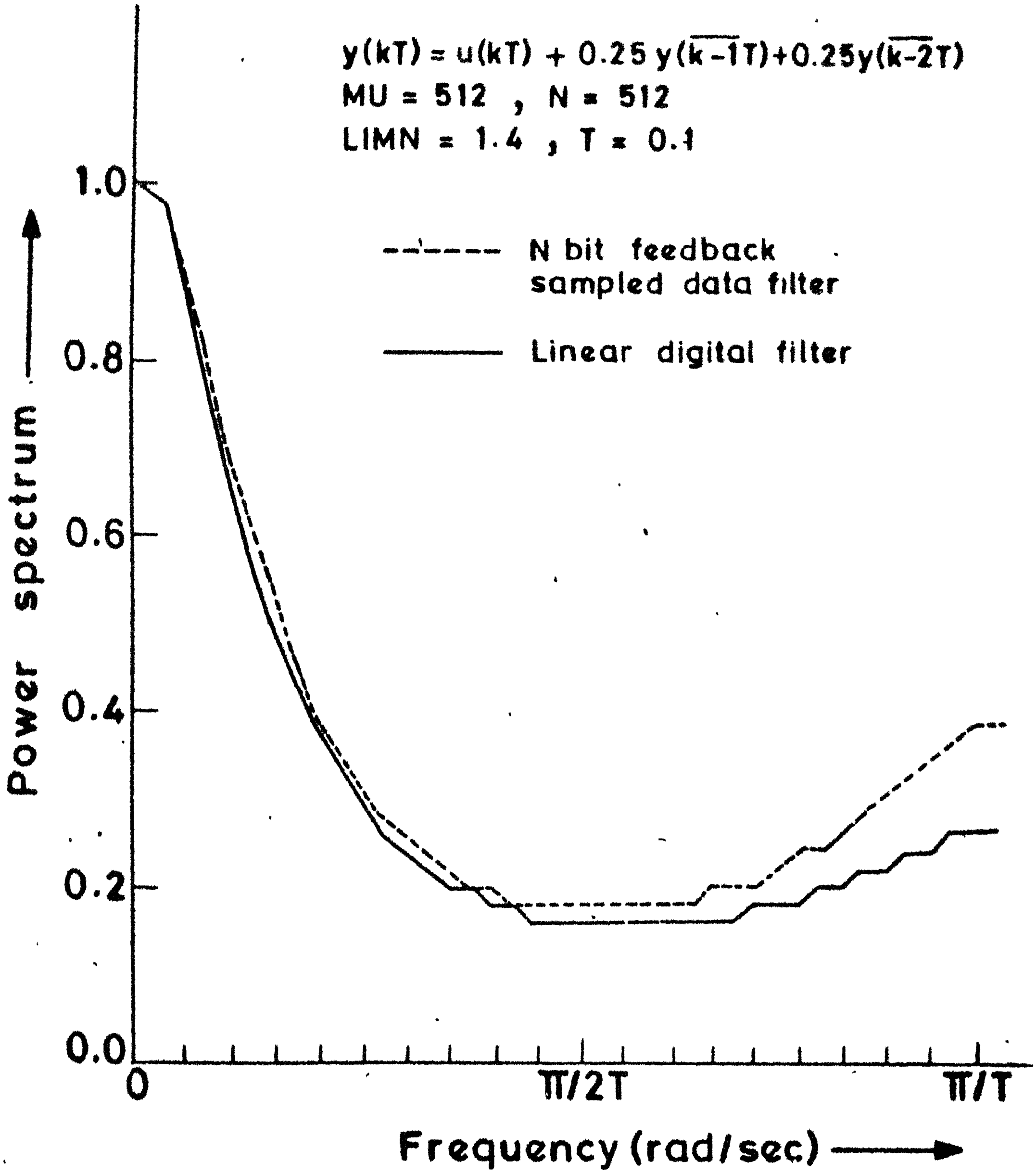


Fig.4.5(a) Second order filters, output power spectra.

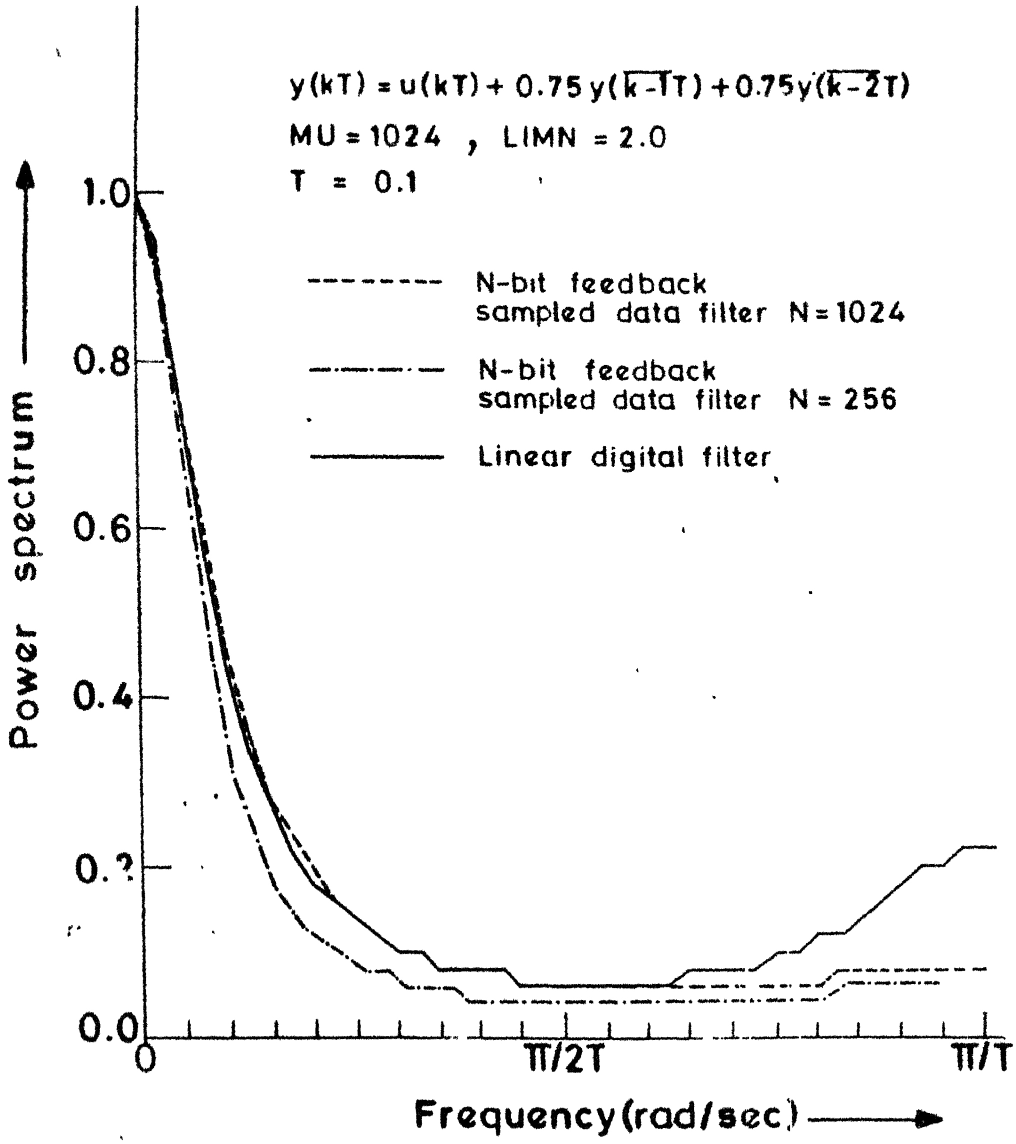


Fig. 4.5(b) Second order filters output power spectra.

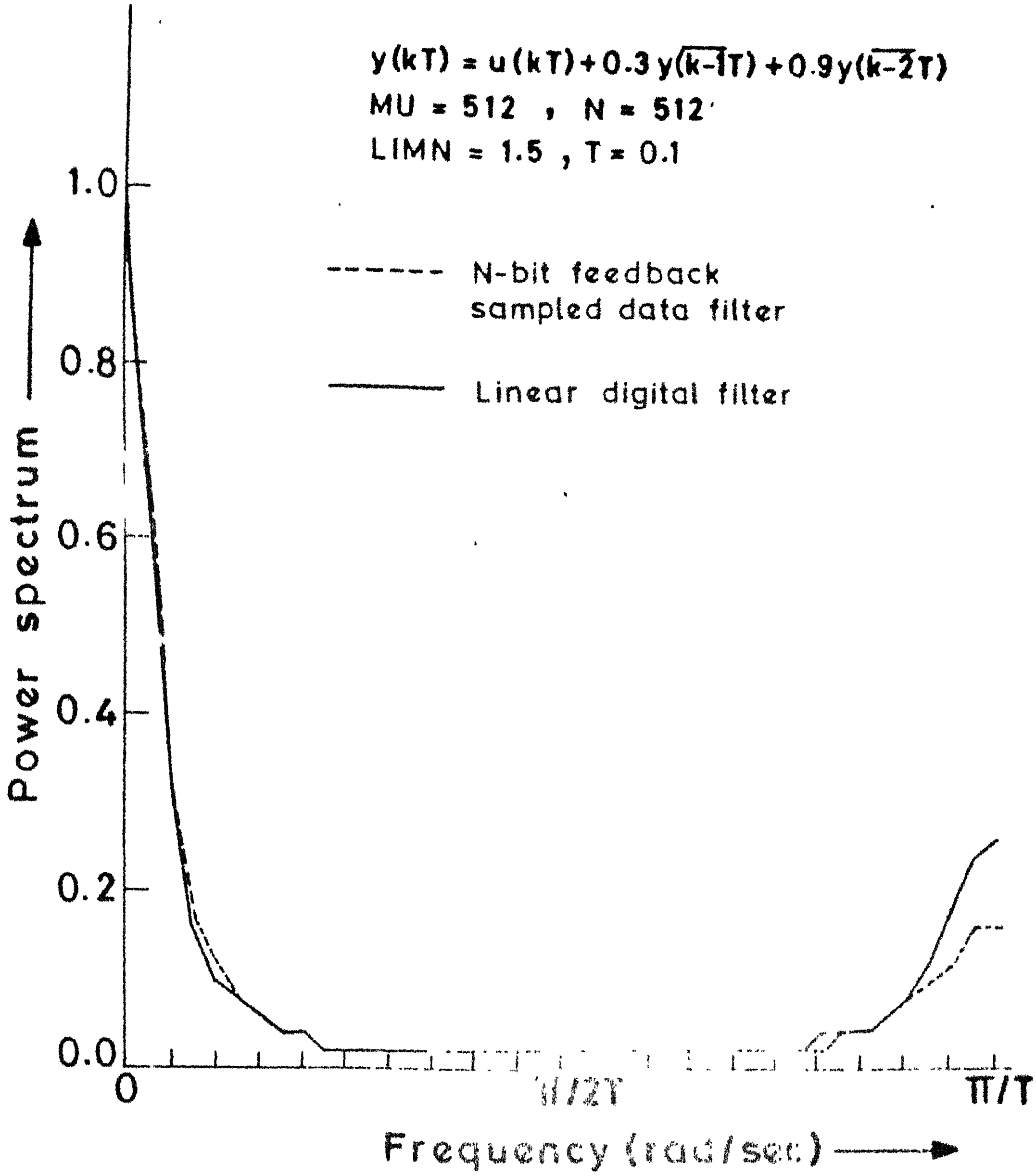


Fig. 4.5(c) Second order filters, output power spectra.

with N was consistent with the explanation given for the "white noise input to first order filters" case. For a fixed MU, varying N over a particular range, the difference between the power spectra of the linear and nonlinear filters is minimum for some N, above and below which the error increases. Fig. 4.5(b) depicts the output power spectra for two different values of N, for a fixed MU.

CHAPTER 5

CONCLUSIONS

In this chapter the results obtained in the preceding study are stated in brief and suggestions for further work on the problem are mentioned.

Linear digital filters are characterized by constant coefficient difference equations. They can be realized using the appropriate units of digital hardware available. The coarse-sampled-data filter was introduced with the idea of replacing the linear digital filter with it. The expected value of the output of the linear digital filter was said to be equal to the expected value of the output of the proposed filter. The advantage was that less complex hardware was required than in the linear digital filter case. However it was noted that due to the requirement of a high frequency dithering signal by the nonlinear filter, the input signals were constrained to those whose Nyquist rates were low.

In Chapter 3, Section 2 the coarse-sampled-data filter using a digital counter averager was investigated. It was found that the output of this filter was equal to that of the linear digital filter in neither its expectation nor its autocorrelation. The coarse-sampled-data filter

structure was modified to yield the N-bit feedback sampled-data filter investigated in Chapter 3, Section 3. It was found that the expected value of the output of the latter filter and that of the output of the linear digital filter were equal. It was also found that the autocorrelation of the output of the N-bit feedback sampled-data filter approached asymptotically the autocorrelation of the output of the linear digital filter as the dithering rate ($1/T' = N/T$) increased, for a fixed input sampling rate ($1/T$). One disadvantage of the N-bit feedback sampled-data filter compared to the coarse-sampled-data filter is that for a linear difference equation of order M, the corresponding coarse-sampled-data filter requires the length of the feedback register to be M, while its modification requires the register length to be MN, where N is the ratio of the input sampling period T to the period of the dither T'.

In Chapter 4, the results obtained by simulating first and second order N-bit feedback sampled-data filters are discussed. It is seen that within a certain amplitude range of the input, the N-bit feedback sampled-data filter responds as a linear filter. The filter requires a minimum 'run time' during which the natural

response due to the initial conditions is allowed to decay to a negligible value, after which the power spectra of the outputs of the N-bit feedback sampled-data filter and the linear digital filter may be compared. It is seen that for the nonlinear filter realised in hardware or simulated on the computer, there is a specific dithering rate for a fixed input sampling rate for which matching of spectra occurs. Below this value of N, the number of points averaged to obtain $\hat{y}(lT)$ is inadequate, and hence the autocorrelation of $\hat{y}(lT)$ is not equal to that of $y(lT)$. Above this optimum value of N, the error introduced due to truncation and/or rounding increases and causes the power spectrum $\hat{y}(lT)$ to increase above that of $y(lT)$.

Scope for Further Work:

The coarse-sampled-data filter proposed by Kirlin has been modified to yield the N-bit sampled-data filter; this uses a 1-bit quantizer. The results may be extended to the case when the quantizer has a greater number of quantization levels. A sensitivity analysis may be carried out to determine the perturbation in the power spectrum due to perturbation of coefficient values.

A requirement of this nonlinear filter is that the input sampling rate be low. Hence the filter may be used

in systems that are multi-rate with $T_{in} > T_{out}$. Covariance Invariant Multirate Filtering requires that an analog filter be replaced by a bank of multirate linear digital filters. The outputs of the individual filters are added after being appropriately delayed, to yield the o/p which is covariance invariant with the o/p of the analog filter. It would be interesting to investigate the power spectrum of the CIMR filter when each of the digital filters in the bank, is replaced by an N-bit feedback sampled-data filter.

As stated previously in Chapter 1, simulators of fading dispersive channels require a great amount of digital hardware. The linear complex filters could be implemented using N-bit feedback coarse-sampled-data filters. A simulation of such a nonlinear complex filter would be highly informative of the nonlinear filter's performance.

APPENDIX : NLF.FOR

PROGRAM FOR THE SIMULATION OF A UNIRATE N-BIT COARSE-SAMPLED-DATA FILTER.

```
      REAL LIMN,LGU,LGY,LGYHAT
      INTEGER RU,RY,RN,X1,XX
      COMPLEX H,V,HH,ANUM,ADEN,R
      DIMENSION U(80),YHAT(80),B(4),A(1),UU(5)
      DIMENSION RFB(2048),RF(0:79),R(0:79)
      DIMENSION PSP(0:79),PSU(0:79),PSY(0:79),PSYHAT(0:79)
      DIMENSION AKK(120)
      DIMENSION Z(4500),TEMP(2048)
      DIMENSION H(0:79),S(0:79)
      DIMENSION LGU(0:79),LGY(0:79),LGYHAT(0:79)
      PI=3.1416
      A1=16.0
      A2=4.0
      READ *,MU,N,RU,RY,TC,LIMN
      READ *,(A(I),I=1,RU)
      READ *,(B(I),I=1,RY)
      RN=N
      M=RU
      YNL=0.0
      MA=MU-M
      ISEED1=67
      X1=57
      DO 460 I=2,MU
      IF(I.GT.2) GO TO 777
      NR=N*RY
      XX=445
      DO 420 J=1,NR
      YY=RAN(XX)
      IF(YY.GE.0.5) GO TO 430
      Z(J)=1.0
      GO TO 420
430   Z(J)=-1.0
420   CONTINUE
777   UX=0.0
      IF(RU.EQ.1) GO TO 44
      DO 55 J=2,RU
      ABC=UU(J)
      UU(J)=UU(1)
      UU(1)=ABC
55    CONTINUE
44    UU(1)=UX
      SU=0.0
      DO 65 J=1,RU
      SU=SU+A(J)*UU(J)
65    CONTINUE
177   ISEED1=ISEED1+8
      CALL GGUB(ISEED1,N,RFB)
      SUM=0.0
      DO 610 II=1,N
      IO=RY
      SZ=0.0
      DO 620 K=1,IO
      KK=K-1
      IP=N*KK+II
      SZ=SZ+B(K)* Z(IP)
620   CONTINUE
      PP=RFB(II)-0.5
      Q=2.0*LIMN*PP
```

A.2

```
      ZHAT=SU+SZ+Q
      IF(ZHAT.GE.0.0)GO TO 27
      TEMP(II)=-1.0
      GO TO 37
27    TEMP(II)=1.0
37    SUM=SUM+TEMP(II)
610   CONTINUE
      IF(KY.EQ.1) GO TO 666
      IR=KY
      DO 630 J=1,N
      DO 640  K=1,IR-1
      IP=N*(IR-K)+J
      IQ=IP-N
      Z(IP)=Z(IQ)
640   CONTINUE
630   CONTINUE
666   DO 550 K=1,N
      Z(K)=TEMP(K)
550   CONTINUE
      IF(I.LE.MA) GO TO 460
      JJ=I-MA
      YHAT(JJ)=SUM*LIMB/RN
      U(IJ)=UX
460   CONTINUE
      GO TO 99
      PI=3.1416
      DO 5 J=0,M-1
      R(J)=U(J+1)
5     CONTINUE
      CALL MEM(R,PSP,IC,M,PI)
      DO 10 I=0,M-1
      PSU(I)=PSP(I)
      LGU(I)=ALOG10(PSU(I))
10    CONTINUE
      DO 810 I=0,M-1
      W=2.0*PI*I/M
      CA=COS(W)
      SINE=SIN(W)
      V=CMPLX(CA,SINE)
      ANUM=V**2
      ADEN=V**2-0.75*V-0.75
      H(I)=ANUM/ADEN
      HH=H(I)
      HH=CONJG(HH)
      S(I)=H(I)*HH
810   CONTINUE
      BIG=S(0)*PSU(0)
      DO 840 I=0,M-1
      P1=S(I)*PSU(I)
      S(I)=P1
      IF(BIG.GE.P1) GO TO 840
      BIG=P1
840   CONTINUE
      DO 850 I=0,M-1
      PSY(I)=S(I)/BIG
      LGY(I)=ALOG10(PSY(I))
850   CONTINUE
      DO 75 J=0,M-1
      R(J)=YHAT(J+1)
75    CONTINUE
      CALL MEM(R,PSP,IC,M,PI)
```

A 3

```
      DO 20 J=0,M-1
      PSYHAT(I)=PSU(I)
      LGYHAT(I)=ALOG10(PSYHAT(I))
20    CONTINUE
      PRINT 105
105   FORMAT(/,11X,'PSU',17X,'PSY',17X,'PSYHAT',14X,'LGU',17X,'LGY',17
     2X,'LGYHAT')
      M=72
      DO 195 J=0,M
      PRINT 175,J,PSU(J),PSY(J),PSYHAT(J),LGU(J),LGY(J),LGYHAT(J)
175   FORMAT(I5,2X,E15.5,5X,E15.5,5X,E15.5,5X,E15.5,5X,E15.5,5X,E15.5)
195   CONTINUE
C TO PLOT LGU,LGY,LGYHAT
      BLANK=' '
      CROSS='#'
      EXCLAM='!'
      ASTER='*'
      ALN1='U'
      ALN2='Y'
      ALN3='Z'
      PP='^'
      PPP='$'
      DO 750 I=0,MM
      DO 670 J=1,80
      AKK(J)=BLANK
670   CONTINUE
      DO 77 J=20,70,10
      AKK(J)=EXCLAM
77    CONTINUE
      AKK(10)=CROSS
      L2=-1
210   IF(L2) 130,140,150
130   AXE=PSU(I)
      GO TO 160
140   AXE=PSY(I)
      GO TO 160
150   AXE=PSYHAT(I)
160   AXE=AXE*50.0
      BB=AINT(AXE)
      BBB=AXE-BB
      IF(ABS(BBB).LT.0.5) GO TO 660
      IF(AXE.GE.0.0)GO TO 110
      BB=-(ABS(BB)+1.0)
      GO TO 660
110   BB=ABS(BB)+1.0
660   IB=BB+10
      IF(L2)170,180,190
170   IBU=IB
      AKK(IBU)=ALN1
      GO TO 200
180   IBY=IB
      IF(IBY.EQ.IBU) GO TO 377
      AKK(IBY)=ALN2
      GO TO 200
377   AKK(IBY)=PP
      GO TO 200
190   IBZ=IB
      IF((IBZ.EQ.IBU.OR.(IBZ.EQ.IBY))GO TO 277
      AKK(IBZ)=ALN3
      GO TO 200
277   AKK(IBZ)=PPP
```

A 4

```
200       L2=L2+1
          IF(L2.LE.1)GO TO 210
          PRINT 680,I,(AKK(J),J=1,70)
680       FORMAT(I3,70A1)
750       CONTINUE
99        STOP
          END
C SUBROUTINE MEM(R,PSP,TC,M,PI)
          SUBROUTINE MEM(R,PSP,TC,M,PI)
          COMPLEX R,T,B,C,PNUM,PDEN,CB,CC,SUMC,TTT,TT,TAP
          COMPLEX ASUM,PT
          DIMENSION RF(0:79),PSP(0:79)
          DIMENSION R(0:79),T(0:79,0:79),B(0:79,0:79),C(0:79,0:79),P(0:79)
          DIMENSION FPE(0:79),TAP(0:79)
          MAXORD=8
          XSUM=0.0
          DO 10 J=0,M-1
          PL=CABS(R(J))
          XSUM=XSUM+PL*PL
10        CONTINUE
          FM=FLOAT(M)
          P(0)=XSUM/FM
          N=1
          DO 20 I=0,M-2
          B(N,I)=R(I)
          C(N,I)=R(I+1)
20        CONTINUE
          DO 225 J=1,MAXORD
          T(J,0)=(1.0,0.0)
225       CONTINUE
100       PNUM=(0.0,0.0)
          PDEN=(0.0,0.0)
          MN=M-1-N
          DO 30 J=0,MN
          CB=CONJG(B(N,J))
          PNUM=PNUM-C(N,J)*CB
          CC=CONJG(C(N,J))
          PDEN=PDEN+B(N,J)*CB+C(N,J)*CC
30        CONTINUE
C CALCULATION OF PREDICTION ERROR COEFFICIENTS
555       T(N,N)=(2.0,0.0)*PNUM/PDEN
          RL=CABS(T(N,N))
          P(N)=P(N-1)*(1.-RL*RL)
          F1=FLOAT(M+N+1)
          F2=FLOAT(M-N-1)
          FPE(N)=P(N)*F1/F2
          IF(N.EQ.1)GO TO 60
          DO 40 J=1,N-1
          TTT=CONJG(T(N-1,N-J))
          T(N,J)=T(N-1,J)+T(N,N)*TTT
40        CONTINUE
```

A.5

```
60        IF(N.EQ.MAXORD)GO TO 1000
          N=N+1
          DO 90 J=0,N-1-1
          T1=CONJG(T(N-1,J-1))
          B(N,J)=B(N-1,J)+T1*C(N-1,J)
          C(N,J)=C(N-1,J+1)+T(N-1,N-1)*B(N-1,J+1)
90        CONTINUE
          GO TO 100
1000      FMIN=0.100E+20
          MINORD=0
          TAP(0)=CMPLX(1.0,0.0)
          DO 500 N=1,MAXORD
          IF(FPE(N).GE.FMIN)GO TO 500
          FMIN=FPE(N)
          MINORD=N
          PMIN=P(N)
500       CONTINUE
CGENERATION OF TAP COEFFICIENTS
          DO 520 J=1,MINORD
          TAP(J)=T(MINORD,J)
520       CONTINUE
          DO 1230 L=0,M-1
          SUMC=(0.0,0.0)
          RF(L)=L/(M*TC)
          DO 1240 K=1,MINORD
          THETA=2.0*PI*RF(L)*K*TC
          TX1=COS(THETA)
          TX2=-SIN(THETA)
          SUMC=SUMC+TAP(K)*CMPLX(TX1,TX2)
1240      CONTINUE
          SUMC=SUMC+(1.0,0.0)
          SAB=CABS(SUMC)
          PSP(L)=PMIN/(SAB**2)
1230      CONTINUE
          BIG=PSP(0)
          DO 870 I=0,M-1
          IF(BIG.GE.PSP(I)) GO TO 870
          BIG=PSP(I)
870       CONTINUE
          DO 880 I=0,M-1
          PSP(I)=PSP(I)/BIG
880       CONTINUE
          RETURN
          END
```

REFERENCES

1. "Theory and Application of Digital Signal Processing" - Rabiner and Gold, PHI Pvt. Ltd. 1978.

2. "Discrete Sampled Data Control Systems" Benjamin Kuo, PH Inc. NJ, 1970.

3. "The Equivalence of Digital and Analog Signal Processing" - K. Steiglitz, Inform. and Control, vol. 8, 1965, pp. 455-467.

4. "Covariance Invariance Digital Filtering" - Perl and Scharf IEEE Trans. ASSP, vol. ASSP - 25, No. 2, Apr. 77, pp. 143-151.

5. "Digital Filtering by Polyphase Network, Application to Sample Rate Alteration and Filter Banks": Bellanger et al IEEE Trans. on ASSP, vol. ASSP-24, No. 2, Apr'76, pp. 109-114.

6. "Covariance - Invariant Multirate Digital Filtering" - K. Sriram, M.Tech. Thesis, Dept. of Electrical. Engg., IIT Kanpur, 1979.

7. "A Statistical Coarse Sampled Data Filter": R.L. Kirlin, IEEE Trans on ASSP, vol. 25, No. 2, Apr'77, pp. 195-196.

8. "Correlation Function Estimation by a Polarity Method using stochastic reference signals" H. Berndt, IEEE Trans. on IT, vol. IT-14, pp. 796-801, Nov'68.

9. "The effect of Dither on luminance quantization of pictures": Lippel and Kurland, IEEE Trans. on Commn. Tech. vol. COM-19, Dec'71, pp. 879-888.

10. "Picture Coding using Pseudo-random noise", L.C. Roberts IRE Trans. on IT, vol. IT-8, pp. 145-154, Feb.'62.

11. "Statistical Analysis of Amplitude Quantized Data Systems", B. Widrow, Trans. of the AIEE Pt. II, and Industry, vol. 79, 1960, pp. 555-568.

12. "[illegible] Quantized Signals", W.R. Bennett, [illegible] [illegible] July'48, [illegible] 146-472.

13. "Digital Processing of Signal[illegible]" [illegible] Gold and [illegible], McGraw Hill, 1969, Chap. 4.

14. "A General Theory of Amplitude Quantization with Applications to Correlation Determination", [illegible] IEE Monograph No. 481M, Nov'61.

15. "Random Process Simulation and Measurements" - G.A. Korn, McGraw Hill, 1966, Chap. 6.

16. "Modified Digital Correlation and its Estim[illegible] Errors", K.Y. Chang and A.D. Moore, IEEE on IT, Nov'70.

17. "A New Method to Compute Correlation Functions" - P. Jespers et al, IRE Trans. IT-8, 1962, p. 106.

18. "A [illegible] Relay Correlation Method", R.E. Bogner, Electronic Letters, vol. I, No. 3, May'65, pp. 53-54.

19. "Correlation Devices and their estimation errors" - J.B. Knowles and H.T. Hsui, Jl. of Appl. P[illegible]s, vol. 38 No. 2, Feb.'67.